Extremal quantum states and combinatorial designs

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Quantum Walk '22

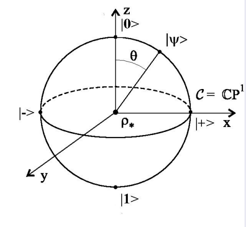
val d'Aosta, June 20, **2022** = 1900+**121**+1

Pure states in a finite dimensional Hilbert space $\mathcal{H}_N$

Qubit = quantum bit; $N = 2$

$$|\psi\rangle = \cos\frac{\vartheta}{2}|1\rangle + e^{i\phi}\sin\frac{\vartheta}{2}|0\rangle$$

Bloch sphere of $N = 2$ pure states (*isotropic*)



Space of pure states for an arbitrary N:

a complex projective space $\mathbb{C}P^{N-1}$ of $2N-2$ real dimensions.

This space is isotropic ! All states are **'equal'** !

There are no states with **extremal** properties !

situation changes if

some structure is imposed to the system

Then all quantum states are **equal**,

but some are **more equal** then others...

a simple example

a composed two-qubit system: $\mathcal{H}_2 \otimes \mathcal{H}_2$
geometric perspective – Segre embedding: $\mathbb{C}P^1 \times \mathbb{C}P^1 \subset \mathbb{C}P^3$.

separable states are distinguished
so are **maximally entangled** **Bell states**, e.g. $|\psi_+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$
as they are **more equal** then other states...

In such a case the search for states with extremal properties is justified:
Bell states are the most entangled *two-qubit* states,
(most distant from the set $\mathbb{C}P^1 \times \mathbb{C}P^1$ of separable states),
useful for several applications in quantum technologies...

Classical Combinatorial Designs

Latin Squares and **Greaco-Latin Squares** are well known subjects considered in *recreational mathematics*.

A **Latin square** of size d is given by d copies of d symbols arranged in a square such that each its row and each column contains different symbols.

cards example of order $d = 3$:

♠	♣	♢
♢	♠	♣
♣	♢	♠

A **Greaco-Latin** square of size d (also called **two orthogonal Latin squares** consists of two Latin squares, (one written with Greek letters one with Latin), such that all d^2 pairs of symbols are different

example of size $d = 3$ studied by **Euler**

αA	βB	γC
γB	αC	βA
βC	γA	αB

Combinatorial design: a constellation of elements of a finite set arranged with certain **symmetry** and **balance**

are related to statistics and planning of experiments

Mutually orthogonal Latin Squares (MOLS)

A classical example:

Take 4 **aces**, 4 **kings**, 4 **queens** and 4 **jacks**
and arrange them into an 4×4 array, such that

a) - in every row and column there is only a **single** card of each suit

b) - in every row and column there is only a **single** card of each rank

A♠	K♣	Q♢	J♥
K♥	A♢	J♣	Q♠
Q♣	J♠	A♥	K♢
J♢	Q♥	K♠	A♣

Two **mutually orthogonal Latin squares** of size $d = 4$

Graeco–Latin square !

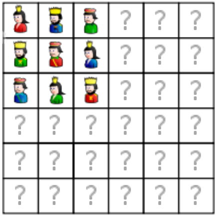
Mutually orthogonal Latin Squares (MOLS)

♣) $d = 2$. There are no orthogonal Latin Square
(for **2** aces and **2** kings the problem has no solution)

♡) $d = 3, 4, 5$ (and any **power of prime**) $\Longrightarrow$ there exist $(d-1)$ MOLS.

♠) $d = 6$. Only a **single** Latin Square exists (No OLS!).

Euler's problem: **36** officers of six different ranks from six different units come for a **military parade**. Arrange them in a square such that in each row / each column all uniforms are different.

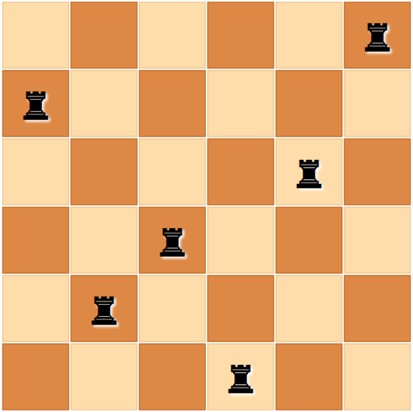


No solution exists ! (**1779** conjecture by **Euler**), proof (**121** years later) **Gaston Tarry** "Le Probléme de 36 Officiers". *Compte Rendu* **(1900)**.

36 officers of Euler revisited

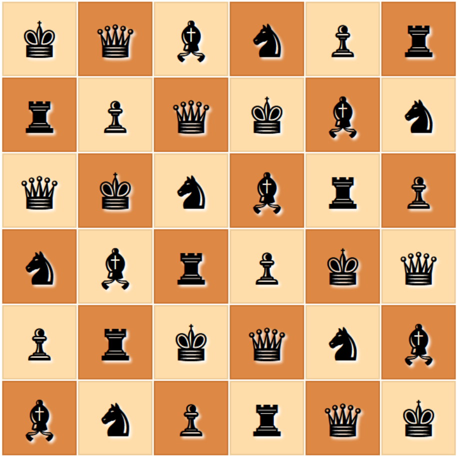
introductory exercise

Step i) Place **six rooks** on a chessboard of size six,
in such a way that no figure attacks any other:



36 officers of Euler, step two

Step ii) Take six pieces of five other figures
and place them onto the board in an analogous way:

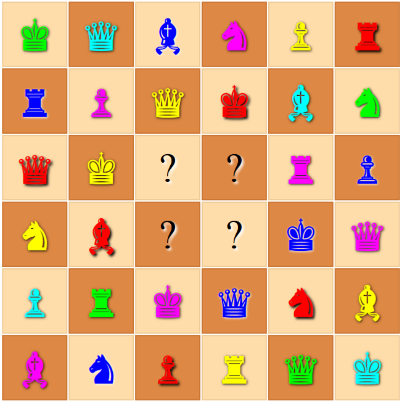


Latin Square of order six

36 officers of Euler, step three...

Step iii) Color them into six **colors**,

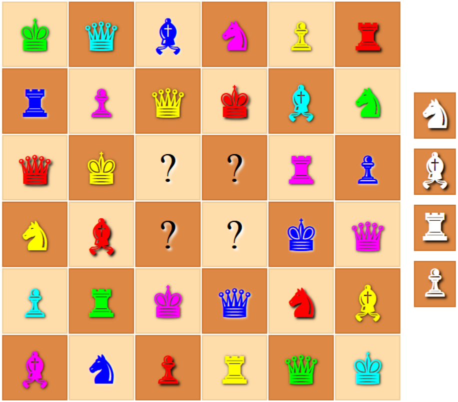
so that all **colors**, in each row and column are different...



Place the remaining four figures, two of them in **cyan** and two in **green**, so that all the rules of Euler are fulfilled

36 officers of Euler, step three $\Rightarrow$ no go !

Step iii) Color them into six **colors**, $d = 6 = 2 * 3$,
so that all **colors**, in each row and column are different...



Place the remaining four figures, two of them in **cyan** and two in **green**, so that all the rules of Euler are fulfilled – **this is not doable !** **G. Tarry**

Quantum Combinatorial Designs

Vicary, Musto (2016): a **square** of order d consisting d^2 states $|\psi_{ij}\rangle \in \mathcal{H}_d$ is called **Quantum Latin** if each of its rows and columns forms an **ortogonal basis**, $\langle\psi_{ij}|\psi_{ik}\rangle = \langle\psi_{ji}|\psi_{ki}\rangle = \delta_{jk}, \ i = 1, \ldots, d$.

Example of order $d = 4$: $\quad QLS(4) \ = \begin{vmatrix} |1\rangle & |2\rangle & |3\rangle & |4\rangle \\ |4\rangle & |3\rangle & |2\rangle & |1\rangle \\ |\chi_-\rangle & |\xi_-\rangle & |\xi_+\rangle & |\chi_+\rangle \\ |\chi_+\rangle & |\xi_+\rangle & |\xi_-\rangle & |\chi_-\rangle \end{vmatrix},$

where $|\chi_\pm\rangle = \frac{1}{\sqrt{2}}(|2\rangle \pm |3\rangle), \ \ |\xi_+\rangle = \frac{1}{\sqrt{5}}(i\,|1\rangle + 2\,|4\rangle)$
and $|\xi_-\rangle = \frac{1}{\sqrt{5}}(2\,|1\rangle + i\,|4\rangle)$ denote **superposition** states,
and give 2 column bases + 2 row bases = 4 **orthogonal bases** in $\mathcal{H}_4$

Standard **combinatorics**: **discrete** set of symbols, $1, 2, \ldots, d$,
+ **permutation** group
generalized **'Quantum' combinatorics**: **continuous** family
of states $|\psi\rangle \in \mathcal{H}_d$ + **unitary** group $U(d)$,
Gerhard Zauner, Ph.D. Thesis, Wien 1999.

Quantum Orthogonal Latin Squares (QOLS)

C) **Classical OLS** consists of d^2 pairs of d symbols such that
a) all d^2 pairs of symbols are **different**,
b,c) there is no **repetition** of any symbol in each row and each column

Q) **Quantum OLS** is formed by d^2 bipartite states

$$|\psi_{ij}\rangle \in \mathcal{H}_d \otimes \mathcal{H}_d = \mathcal{H}_A \otimes \mathcal{H}_B \quad \text{such that}$$

a)' all d^2 states are **mutually orthogonal**, $\langle\psi_{ij}|\psi_{k\ell}\rangle = \delta_{ik}\delta_{j\ell}$,
b)', c)' all rows and columns satisfy **partial trace** relations

$$\mathrm{Tr}_B\left(\sum_{k=0}^{d-1} |\psi_{ik}\rangle\langle\psi_{jk}|\right) = \delta_{ij}\mathbb{I}_d$$

$$\mathrm{Tr}_B\left(\sum_{k=0}^{d-1} |\psi_{ki}\rangle\langle\psi_{kj}|\right) = \delta_{ij}\mathbb{I}_d$$

and dual conditions for Tr_A.

a 'classical' $d = 3$ example of **QOLS**:

$\vert A\spadesuit\rangle$	$\vert K\clubsuit\rangle$	$\vert Q\diamondsuit\rangle$
$\vert K\diamondsuit\rangle$	$\vert Q\spadesuit\rangle$	$\vert A\clubsuit\rangle$
$\vert Q\clubsuit\rangle$	$\vert A\diamondsuit\rangle$	$\vert K\spadesuit\rangle$

is based on **product** states, e.g. $|K\clubsuit\rangle = |K\rangle \otimes |\clubsuit\rangle$.
To get **genuinely Quantum OLS** one needs to introduce **entanglement**

Composed systems & entangled quantum states

bi-partite systems: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

- **separable pure states:** $|\psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$
- **entangled pure states:** all states **not** of the above product form.

Two–qubit system: $2 \times 2 = 4$

Maximally entangled Bell state $|\varphi^+\rangle := \Big(|00\rangle + |11\rangle\Big) = (|A♠\rangle + |K♣\rangle)$

useful for several applications in quantum technologies...

Maximally Entangled states of a $d \times d$ system

Any pure state from $\mathcal{H}_d \otimes \mathcal{H}_d$ can be written as

$|\psi\rangle = \sum_{ij} C_{ij} |i\rangle \otimes |j\rangle$, where $|\psi|^2 = \mathrm{Tr}\, CC^\dagger = 1$.

A state $|\psi\rangle$ is **maximally entangled** if its partial trace is maximally mixed,

$\sigma = \mathrm{Tr}_B |\psi\rangle\langle\psi| = CC^\dagger = \mathbb{1}_d / d$,

which is the case if the rescaled **matrix** $U = \sqrt{d}C$ of size d is **unitary**.

Absolutely maximally entangled state (AME)

Definition. A pure state of an even number N of qu*d*its is called **absolutely maximally entangled**, **AME(N,d)** if for any choice of $N/2$ subsystems traced out the reduced state is maximally mixed.

Scott (2004), **Facchi**+ (2008), **Helwig**+ (2012), **Arnaud**+ (2013)

An **AME state** of four parties A, B, C, D with d levels each,

$$|\psi\rangle = \sum_{i,j,l,m=1}^{d} T_{ijlm} |i,j,l,m\rangle$$

It is **maximally entangled** with respect to all **three** partitions:

$$AB|CD \text{ and } AC|BD \text{ and } AD|BC.$$

Let $\rho_{ABCD} = |\psi\rangle\langle\psi|$. Hence its three reductions are **maximally mixed**,

$$\rho_{AB} = \mathrm{Tr}_{CD}\rho_{ABCD} = \rho_{AC} = \mathrm{Tr}_{BD}\rho_{ABCD} = \rho_{AD} = \mathrm{Tr}_{BC}\rho_{ABCD} = \mathbb{1}_{d^2}/d^2$$

Thus matrices $U_{\mu,\nu}$ of order d^2 obtained by reshaping the tensor T_{ijkl} are **unitary** for three reorderings:

$$\text{a) } \mu,\nu = ij, lm, \quad \text{b) } \mu,\nu = im, jl, \quad \text{c) } \mu,\nu = il, jm.$$

Such a tensor T is called **perfect**, **Pastawski** et al. (2015), and a matrix U **two–unitary** **Goyeneche** et al. (2015)

AME states and Quantum OLS

Theorem. Existence of QOLS(d) is equivalent to AME states of 4 systems with d levels each.

each field of a QOLS(d) encodes four data: two digits from one to d determine address of a square, two other data its content.

Let $\{|\phi_{ij}\rangle \in \mathcal{H}_d\}_{i,j=1}^{d}$ form a QOLS(d).
Then 4-partite state $|\Psi_4\rangle := \sum_{i,j=1}^{d} |i,j\rangle \otimes |\phi_{ij}\rangle = \sum_{i,j,k,\ell=1}^{d} t_{ijk\ell}|i,j,k,\ell\rangle$ forms the state $|AME(4,d)\rangle$ while $t_{ijk\ell}$ forms a **perfect tensor** as reshaped into a matrix $U_{\mu\nu}$ is unitary for all pairs of indices;

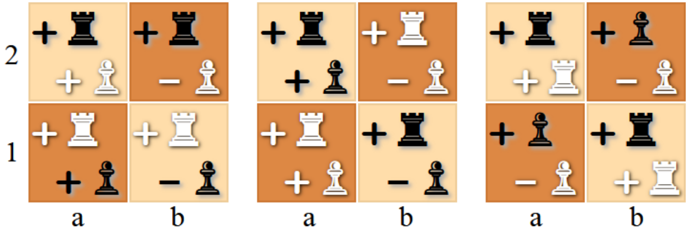
$$\mu = (i,j);\ \mu = (i,k);\ \mu = (i,\ell).$$

Maximal entanglement of a four-party state $|\Psi_4\rangle = |\Psi\rangle_{ABCD}$
with respect to all **three** partitions: $AB|CD$ and $AC|BD$ and $AD|BC$
is equivalent to the fact that d^2 bi-partite states $|\phi_{ij}\rangle$
form a **quantum** **ortogonal Latin square**.

No Quantum OLS of order $d = 2$

There are no **classical OLS** of size $d = 2$

There are no **quantum OLS** of size $d = 2$ either!



Even using entangled states (*more than a single figure in one chess field*) it is not possible to find a 2×2 square of four states which satisfies QOLS conditions **a'), b'), c')**.

Higuchi, Sudbery (2001) proved that there are no AME states of 4 qubits $\Longrightarrow$ no QOLS(2) !

Higher dimensions: AME(4,3) state of four qutrits

A **Greaco-Latin square** of size $d = 3$

each symbol encodes 4 digits: (c,r,f,s) = column, row, figure, suit

αA	βB	γC
γB	αC	βA
βC	γA	αB

=

$A\spadesuit$	$K\clubsuit$	$Q\diamondsuit$
$K\diamondsuit$	$Q\spadesuit$	$A\clubsuit$
$Q\clubsuit$	$A\diamondsuit$	$K\spadesuit$

=

0, 0	1, 2	2, 1
1, 1	2, 0	0, 2
2, 2	0, 1	1, 0

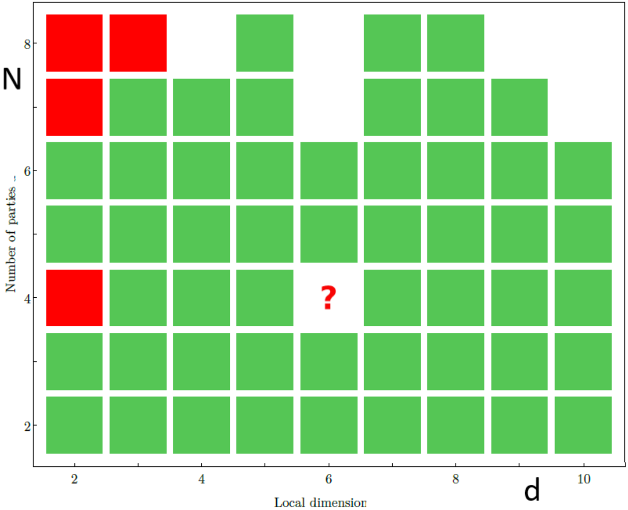
yields an **AME state** of **4 qutrits**:

$$|\Psi_3^4\rangle = |0000\rangle + |0112\rangle + |0221\rangle + |1011\rangle + |1120\rangle + |1202\rangle + |2022\rangle + |2101\rangle + |2210\rangle.$$

Corresponding **Quantum Code**:

$$|0\rangle \to |\tilde{0}\rangle := |000\rangle + |112\rangle + |221\rangle$$
$$|1\rangle \to |\tilde{1}\rangle := |011\rangle + |120\rangle + |202\rangle$$
$$|2\rangle \to |\tilde{2}\rangle := |022\rangle + |101\rangle + |210\rangle$$

Existence of **Absolutely maximally entangled** states



The case: $N = 4$ subsystems with $d = 6$ levels each

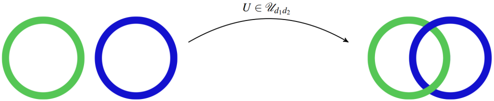
(*corresponding to 36 officers of Euler*) up till 2021

remained open! [1]see on-line table by **F. Huber & N. Wyderka**

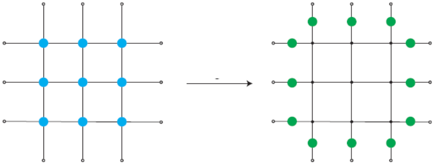
In hunt for an $|AME(4,6)\rangle$ state of 4 quhex

To find the state
$|AME(4,6)\rangle = (U_{AB} \otimes \mathbb{I}_{CD})|\Psi^+_{AC|BD}\rangle = \sum_{i,j,k,\ell=1}^{6} t_{ijk\ell}|i,j,k,\ell\rangle$

we look for a 2-unitary matrix $U_{AB} \in U(36)$, which remains unitary after reorderings, maximizes the **entangling power** $e_p(U)$

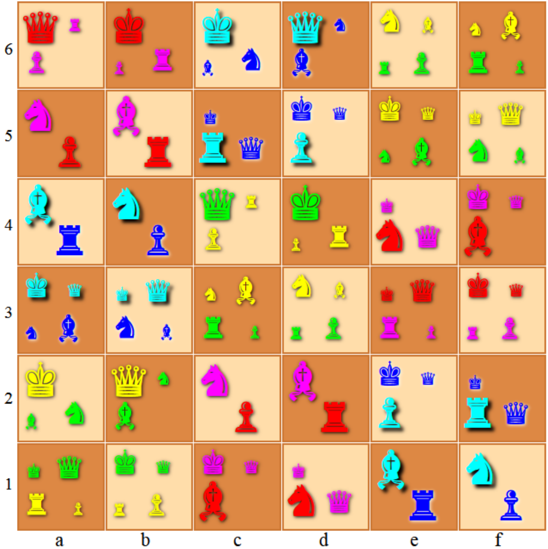


(average entanglement of $U_{AB}|\psi_A\rangle \otimes |\psi_B\rangle$)
and leads to a perfect tensor $t_{ijk\ell}$ used for models of bulk/boundary duality

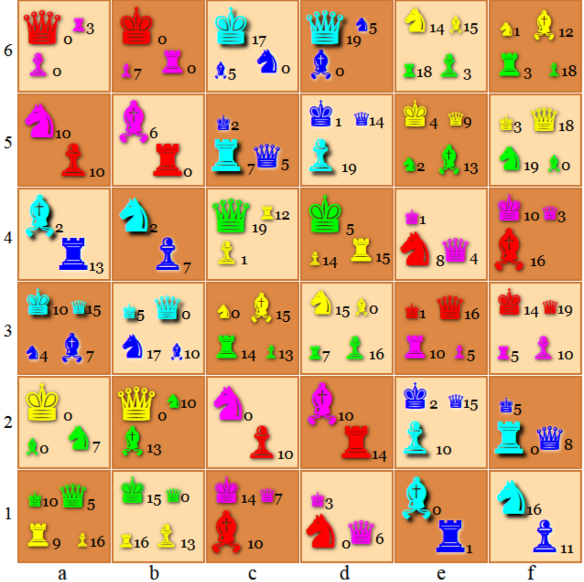


Optimization over the space $U(36)$ of dimension $36^2 - 1 = 1295$
is not easy...

No classical OLS(6). **But a quantum solution exists !**



Quantum solution of *36 entangled officers* of **Euler**. Size of the figures represents moduli of superpositions, Is field $c2$ equal to $a5$?



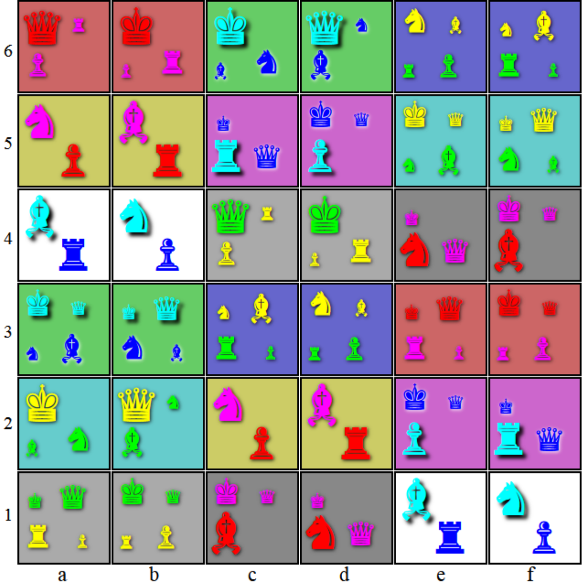
Quantum solution of *36 entangled officers* of **Euler**. Size of the figures represents moduli of superpositions, index k denotes the complex **phase** $\exp(i\pi k/20)$, e.g. field c2) denotes |♞⟩ − |♟⟩ and is orthogonal to a5).

Full solution of the problem of 36 **entangled officers of Euler** encoded in the chessboard of size 6 looks like this...
(each state $|\psi_{ij}\rangle$ determines a single row of a 2-unitary matrix U_{36})

$$|\psi_{00}\rangle = c|10\rangle + a\omega^3|43\rangle + b|53\rangle = c|\text{♛}\rangle + a\omega^3|\text{♜}\rangle + b|\text{♟}\rangle$$
$$|\psi_{01}\rangle = c|00\rangle + b|43\rangle + a\omega^7|53\rangle = c|\text{♚}\rangle + b|\text{♜}\rangle + a\omega^7|\text{♟}\rangle$$
$$|\psi_{02}\rangle = c\omega^{17}|01\rangle + b|24\rangle + a\omega^5|34\rangle = c\omega^{17}|\text{♚}\rangle + b|\text{♞}\rangle + a\omega^5|\text{♝}\rangle$$
$$|\psi_{10}\rangle = c\omega^{10}|23\rangle + c\omega^{10}|50\rangle = c\omega^{10}|\text{♞}\rangle + c\omega^{10}|\text{♟}\rangle$$
$$|\psi_{11}\rangle = c\omega^{6}|33\rangle + c|40\rangle = c\omega^{6}|\text{♝}\rangle + c|\text{♜}\rangle$$
$$|\psi_{12}\rangle = a\omega^{2}|04\rangle + b\omega^5|14\rangle + c\omega^7|41\rangle = a\omega^{2}|\text{♚}\rangle + b\omega^5|\text{♛}\rangle + c\omega^7|\text{♜}\rangle$$
$$\ldots \quad = \quad \ldots$$
$$|\psi_{55}\rangle = c\omega^{16}|21\rangle + c\omega^{11}|54\rangle = c\omega^{16}|\text{♞}\rangle + c\omega^{11}|\text{♟}\rangle,$$

where $\omega = \exp(i\pi k/20)$, and $a^2 + b^2 = c^2 = 1/2$,
while the ratio of the two sizes of the figures is equals to the
golden mean, $b/a = (1+\sqrt{5})/2 = \varphi$.

It is easy to check that this constellation satisfies the desired conditions **a'), b'), c')** specified above and it deservs an appelation golden **square**.



Four states on background of the same colour form a basis and are **orthogonal** ! The board of size 6 with 36 fields is divided into 9 groups of 4 two-qubit orthogonal states. **9*4=6*6**

ENTANGLING POWER

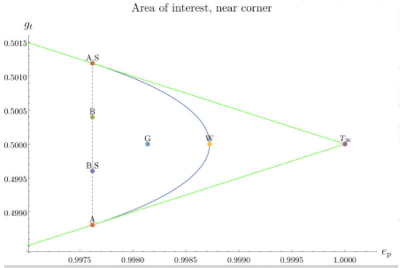
$$e_p(U) = \frac{1}{E(S)}\Big(E(U) + E(US) - E(S)\Big) \in [0,1]$$

$$E(U) = 1 - \big(\textstyle\sum_{j=1}^{d^2} \lambda_j^2\big)/d^4$$

$$g_t(U) = \frac{1}{2E(S)}\Big(E(U) - E(US) + E(S)\Big) \in [0, \tfrac{1}{2}]$$

$$S\Big(|\phi_A\rangle \otimes |\phi_B\rangle\Big) = |\phi_B\rangle \otimes |\phi_A\rangle$$

$U \in \mathcal{U}_{d^2}$ is 2-unitary $\Leftrightarrow$ $E(U) = E(SU) = E(S)$ $\Leftrightarrow$ $e_p(U) = 1, g_t(U) = \frac{1}{2}$



11	22	33	44	55	66
23	14	45	36	61	52
32	41	64	53	16	25
46	35	51	62	24	13
54	63	26	15	42	31
65	56	12	21	33	44

=

A♠	K♣	Q♦	J♥	10✿	9✷
K♦	A♥	J✿	Q✷	9♠	10♣
Q♣	J♠	9♥	10♦	A✷	K✿
J✷	Q✿	10♠	9♣	K♥	A♦
10♥	9♦	K✷	A✿	J♣	Q♠
9✿	10✷	A♣	K♠	Q♦	J♥

$$e_p = \frac{314}{315}$$

Numerical Search

$$U_0 \mapsto U_0^{\mathrm{R}} \mapsto \left(U_0^{\mathrm{R}}\right)^{\Gamma} := U_0^{\Gamma\mathrm{R}} \mapsto U_1$$

$\tilde{P} =$

11	22	33	44	55	66
23	14	45	36	61	52
32	41	64	53	16	25
46	35	51	62	24	13
54	63	26	15	42	31
65	56	12	21	33	44

=

A♠	K♣	Q♦	J♥	10✿	9✱
K♦	A♥	J✿	Q✱	9♠	10♣
Q♣	J♠	9♥	10♦	A✱	K✿
J✱	Q✿	10♠	9♣	K♥	A♦
10♥	9♦	K✱	A✿	J♣	Q♠
9✿	10✱	A♣	K♠	Q♦	J♥

$$e_p(\tilde{P}) = \frac{314}{315} \simeq .9968 \qquad e_p(\tilde{P}e^{iH\varepsilon}) \to .9991$$

$$e_p(\tilde{P}) \to \frac{419}{420} \simeq .9976$$

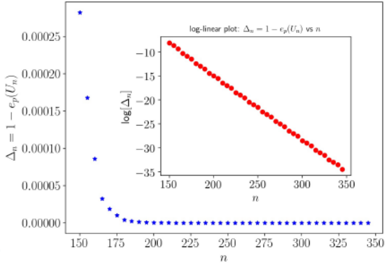
$$e_p(\widetilde{P_s}) = \frac{104}{105} \simeq .9905 \qquad e_p(\widetilde{P_S}e^{iH\varepsilon}) \to 1$$

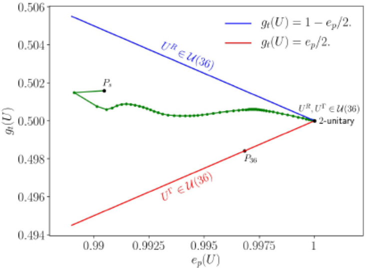
$\tilde{P}_s =$

11	22	33	44	55	66
23	14	45	36	61	52
32	41	64	53	16	25
46	35	51	62	24	13
64	56	26	15	43	31
55	63	12	21	42	34

=

A♠	K♣	Q♦	J♥	10✿	9✱
K♦	A♥	J✿	Q✱	9♠	10♣
Q♣	J♠	9♥	10♦	A✱	K✿
J✱	Q✿	10♠	9♣	K♥	A♦
9♥	10✱	K✱	A✿	J♦	Q♠
10✿	9♦	A♣	K♠	J♣	Q♥

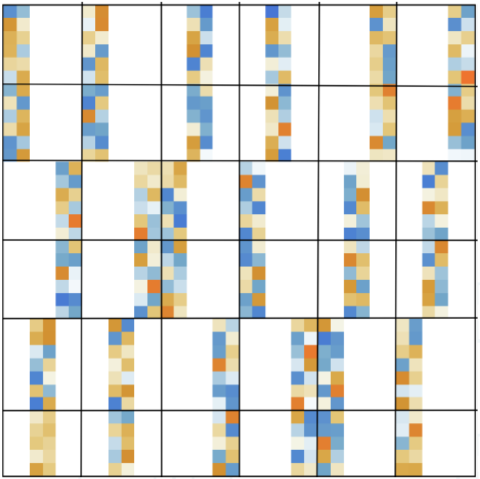




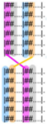
Numerical Cleaning

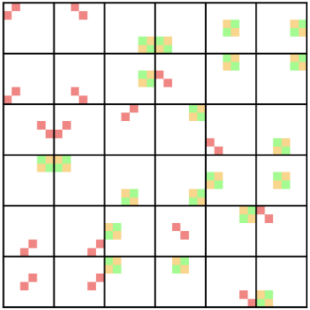
$$\left(U_6^{(1)} \otimes U_6^{(2)}\right) U_{36} \left(U_6^{(4)} \otimes U_6^{(3)}\right)$$

$$\left(\cancel{U_6} \otimes U_2^{\otimes 3}\right) U_{36} \left(U_2^{\otimes 3} \otimes U_2^{\otimes 3}\right)$$

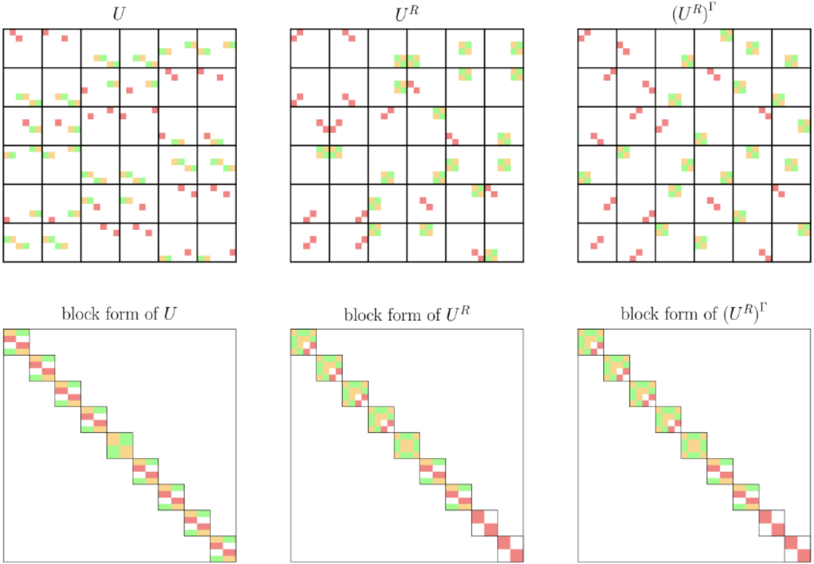


0.75	0	0	0	0	0	0.25	0
0	0.25	0	0	0	0	0	0.25
0	0	0.25	0	0.25	0	0	0
0	0	0	0.75	0	0.25	0	0
0	0	0.25	0	0.25	0	0	0
0	0	0	0.25	0	0.75	0	0
0.25	0	0	0	0	0	0.75	0
0	0.25	0	0	0	0	0	0.25





Solution Found



Solution Found

(1,1)	(2,2)	(1,2)	(2,1)	
$a\,\omega^{10}$	a	$b\,\omega^{15}$	$b\,\omega^{5}$	(6,3)
		c	c	(1,1)
c	c			(5,6)
$b\,\omega^{10}$	b	$a\,\omega^{5}$	$a\,\omega^{15}$	(4,2)

(1,3)	(2,4)	(1,4)	(2,3)	
$a\,\omega^{2}$	$a\,\omega^{14}$	$b\,\omega$	$b\,\omega^{5}$	(2,5)
		$c\,\omega^{5}$	$c\,\omega^{19}$	(3,3)
$c\,\omega^{17}$	$c\,\omega^{19}$			(1,2)
$b\,\omega^{14}$	$b\,\omega^{6}$	$a\,\omega^{3}$	$a\,\omega^{7}$	(6,4)

(1,5)	(2,6)	(1,6)	(2,5)	
$a\,\omega$	$a\,\omega^{19}$	$b\,\omega^{14}$	$b\,\omega^{16}$	(4,1)
$a\,\omega$	$a\,\omega^{3}$	$b\,\omega^{10}$	$b\,\omega^{4}$	(3,4)
$b\,\omega^{4}$	$b\,\omega^{18}$	$a\,\omega^{3}$	$a\,\omega^{9}$	(2,6)
$b\,\omega^{2}$	$b\,\omega^{8}$	$a\,\omega^{5}$	$a\,\omega^{15}$	(5,5)

(3,1)	(4,2)	(3,2)	(4,1)	
$a\,\omega^{4}$	$a\,\omega^{10}$	$b\,\omega^{17}$	$b\,\omega^{7}$	(4,5)
		$c\,\omega^{2}$	$c\,\omega^{2}$	(3,2)
$c\,\omega^{10}$	$c\,\omega^{6}$			(2,4)
$b\,\omega^{7}$	$b\,\omega^{13}$	$a\,\omega^{10}$	a	(5,3)

(3,3)	(4,4)	(3,4)	(4,3)	
a	a	$b\,\omega^{15}$	$b\,\omega^{15}$	(4,6)
		c	$c\,\omega^{10}$	(6,1)
c	$c\,\omega^{10}$			(5,4)
b	b	$a\,\omega^{5}$	$a\,\omega^{5}$	(1,5)

(3,5)	(4,6)	(3,6)	(4,5)	
$a\,\omega^{2}$	a	$b\,\omega^{19}$	$b\,\omega^{13}$	(2,3)
		$c\,\omega^{16}$	c	(6,2)
$c\,\omega^{8}$	$c\,\omega^{16}$			(3,1)
$b\,\omega^{14}$	$b\,\omega^{12}$	$a\,\omega$	$a\,\omega^{15}$	(1,6)

(5,1)	(6,2)	(5,2)	(6,1)	
$a\,\omega^{3}$	$a\,\omega^{7}$	b	b	(1,4)
		c	$c\,\omega^{10}$	(2,1)
$c\,\omega^{13}$	$c\,\omega^{7}$			(3,5)
$b\,\omega^{9}$	$b\,\omega^{13}$	$a\omega^{16}$	$a\omega^{16}$	(6,6)

(5,3)	(6,4)	(5,4)	(6,3)	
$a\,\omega^{12}$	$a\,\omega^{14}$	$b\,\omega^{15}$	$b\,\omega$	(3,6)
		$c\,\omega^{14}$	$c\,\omega^{10}$	(5,1)
$c\,\omega^{7}$	$c\,\omega^{19}$			(2,2)
$b\,\omega^{14}$	$b\,\omega^{16}$	$a\omega^{7}$	$a\omega^{13}$	(4,3)

(5,5)	(6,6)	(5,6)	(6,5)	
$a\,\omega^{18}$	$a\,\omega^{18}$	$b\,\omega^{3}$	$b\,\omega^{3}$	(1,3)
		c	$c\,\omega^{10}$	(5,2)
$c\,\omega$	$c\,\omega^{11}$			(6,5)
$b\,\omega^{10}$	$b\,\omega^{10}$	$a\omega^{5}$	$a\omega^{5}$	(4,4)

AME(4,6) state

$$\frac{1}{6}\sum_{i,j,k,\ell=1}^{d} t_{i,j,k,\ell}\,|i\rangle\,|j\rangle\,|k\rangle\,|\ell\rangle$$

$$a = \frac{1}{\sqrt{2}(\omega+\overline{\omega})} = \frac{1}{\sqrt{5+\sqrt{5}}}$$

$$b = \frac{1}{\sqrt{2}(\omega^3+\overline{\omega}^3)} = \sqrt{\frac{5+\sqrt{5}}{20}}$$

$$c = \frac{1}{\sqrt{2}}$$

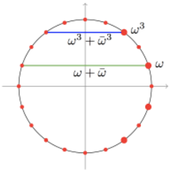
$$\omega = \exp(i\,\pi/20)$$

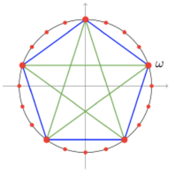
Pythagoras theorem

$$a^2 + b^2 = c^2 = \frac{1}{2}$$

Golden ratio

$$b/c = \varphi = = \frac{1+\sqrt{5}}{2}$$





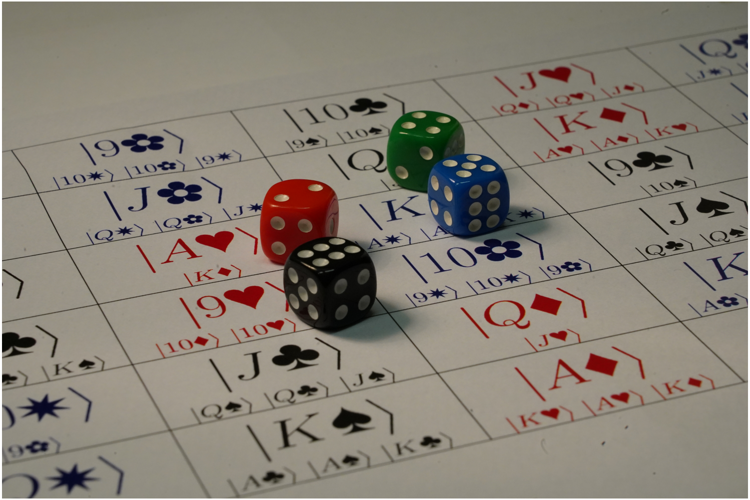
Quantum Officers of Euler

|A♣⟩ |K♠⟩	|A♦⟩ |K♥⟩	|10✿⟩ |10✱⟩ |9✿⟩ |9✱⟩	|10♠⟩ |10♣⟩ |9♠⟩ |9♣⟩	|Q♦⟩ |Q♥⟩ |J♦⟩ |J♥⟩	|Q✿⟩ |Q✱⟩ |J✿⟩ |J✱⟩
|10♣⟩ |9♠⟩	|10♦⟩ |9♥⟩	|Q✿⟩ |Q✱⟩ |J✿⟩ |J✱⟩	|Q♠⟩ |J♣⟩	|A♦⟩ |A♥⟩ |K♦⟩ |K♥⟩	|A✿⟩ |A✱⟩ |K✿⟩ |K✱⟩
|Q✿⟩ |J✱⟩	|Q♣⟩ |J♠⟩	|A♥⟩ |K♦⟩	|A✿⟩ |A✱⟩ |K✿⟩ |K✱⟩	|10♠⟩ |9♣⟩	|10♦⟩ |10♥⟩ |9♦⟩ |9♥⟩
|A✿⟩ |A✱⟩ |K✿⟩ |K✱⟩	|A♠⟩ |A♣⟩ |K♠⟩ |K♣⟩	|10♦⟩ |10♥⟩ |9♦⟩ |9♥⟩	|10✿⟩ |10✱⟩ |9✿⟩ |9✱⟩	|Q♠⟩ |Q♣⟩ |J♠⟩ |J♣⟩	|Q♦⟩ |Q♥⟩ |J♦⟩ |J♥⟩
|10♥⟩ |9♦⟩	|10✱⟩ |9✿⟩	|Q♠⟩ |Q♣⟩ |J♠⟩ |J♣⟩	|Q♦⟩ |J♥⟩	|A✿⟩ |A✱⟩ |K✿⟩ |K✱⟩	|A♠⟩ |K♣⟩
|Q♥⟩ |J♦⟩	|Q✱⟩ |J✿⟩	|A♠⟩ |A♣⟩ |K♠⟩ |K♣⟩	|A♦⟩ |A♥⟩ |K♦⟩ |K♥⟩	|10✿⟩ |9✱⟩	|10♠⟩ |10♣⟩ |9♠⟩ |9♣⟩

$A/K \rightarrow A$
$D/J \rightarrow B$
$10/9 \rightarrow C$

♠/♣ → α
♦/♥ → β
✿/✱ → γ

$A\alpha$	$A\beta$	$C\gamma$	$C\alpha$	$B\beta$	$B\gamma$
$C\alpha$	$C\beta$	$B\gamma$	$B\alpha$	$A\beta$	$A\gamma$
$B\gamma$	$B\alpha$	$A\beta$	$A\gamma$	$C\alpha$	$C\beta$
$A\gamma$	$A\alpha$	$C\beta$	$C\gamma$	$B\alpha$	$B\beta$
$C\beta$	$C\gamma$	$B\alpha$	$B\beta$	$A\gamma$	$A\alpha$
$B\beta$	$B\gamma$	$A\alpha$	$A\beta$	$C\gamma$	$C\alpha$



Four dice in the golden $|AME(4,6)\rangle$ state corresponding to *36 entangled officers* of **Euler**. Any pair of dice is unbiased, although their outcome determines the state of the other two.

Concluding Remarks

Strongly entangled extremal **multipartie** quantum states can be useful for quantum error correction codes, multiuser quantum communication and other protocols.

Theorem. Absolutely maximally entangled states $|AME(4,6)\rangle$ of 4 subsystems with 6 levels each **do** exist !

Rather, Burchardt, Bruzda, Rajchel, Lakshminarayan, K.Ż. preprint arXiv:2102.07787, *April 2021*. (**121** years after Tarry) and *Phys. Rev. Lett.* (2022). This implies **existence** of

1. solution of the quantum analogue of the 36 officers problem of **Euler**,
2. optimal bi-partite unitary gate U_{36} with maximal **entangling power**
3. **perfect tensor** $t_{ijk\ell}$ with 4 indices, each running from 1 to 6, to be applied for tensor networks and bulk/boundary correspondence,
4. nonadditive **quantum error correction code** $((3,6,2))_6$ which allows one to encode a single quhex in three quhexes
5. distinguished point in $\mathbb{C}P^{36\times 36-1} \supset \mathbb{C}P^5 \times \mathbb{C}P^5 \times \mathbb{C}P^5 \times \mathbb{C}P^5$

$\Longrightarrow$ such quantum states with *extremal* properties can be useful...

Thirty-six entangled officers[1] of Euler, $|\psi_{13}\rangle = (|♚\rangle + |♛\rangle)/\sqrt{2}$

[1] It is sad to note that these Russian officers recently left their parade ground in Saint Petersburg, where they belong, and went a thousand miles South...

However, explicit analytical results described in this work strongly suggest that the officers might eventually suffer a transition

into a highly *entangled* state.