On resource theories for sets of quantum measurements

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- Reminder: resource theories for quantum states

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- Sets of quantum measurements: resource quantification

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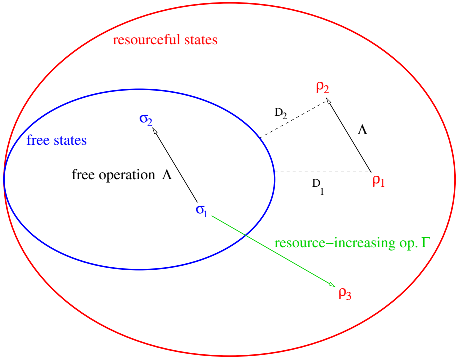
L. Tendick, M. Kliesch, H. Kampermann, and DB, arXiv:2205.08546

Reminder: Resource theories for quantum states

E. Chitambar and G. Gour, Rev. Mod. Phys. 91, 025001 (2019)



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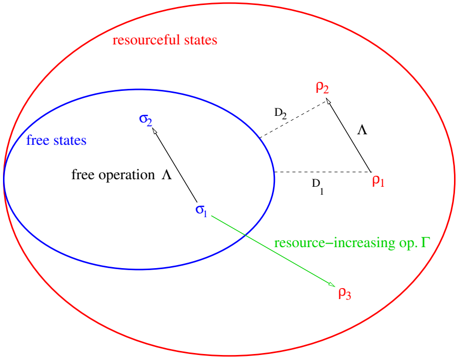


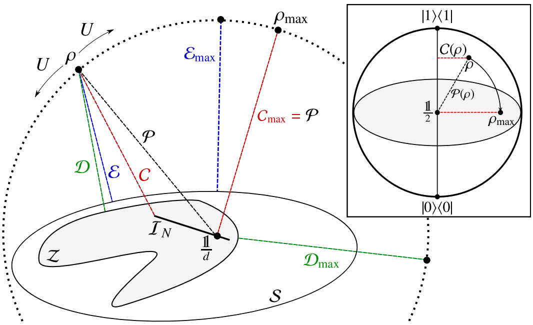
Example for convex quantum resource theory: Entanglement

1) Free states: separable; resourceful states: entangled
2) Free operations: LOCC; resource operations: entanglement-creating
3) Quantification: distance measures, robustness measures, concurrence...

Purity as bound for coherence, discord and entanglement

Let purity $P(\rho)$, coherence $C(\rho)$, discord $\mathcal{D}(\rho)$, and entanglement $\mathcal{E}(\rho)$ be distance-based measures for ρ. Then the following hierarchy holds, with $\mathcal{X}_{\mathsf{max}}(\rho) = \max_U \mathcal{X}(U\rho U^\dagger)$:

$$P(\rho) = C_{\mathsf{max}}(\rho) \geq \mathcal{D}_{\mathsf{max}}(\rho) \geq \mathcal{E}_{\mathsf{max}}(\rho)$$



A. Streltsov, H. Kampermann, S. Wölk, M. Gessner, and DB, New J. Phys. 20, 053058 (2018)

Reminder: Quantum measurement

Positive operator valued measurement (POVM):

Set $\mathcal{M} = \{M_a\}$, with POVM elements $M_a \geq 0$ and $\sum_a M_a = \mathbb{1}_d$,
acting on d-dim Hilbert space $\mathcal{H}_d$,
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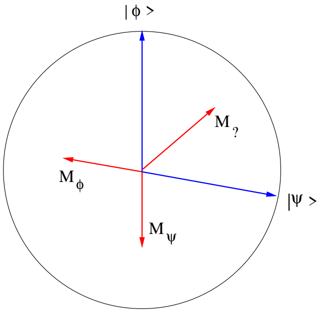
Example: Unambiguous State Discrimination



Sets of measurements ($\equiv$ measurement assemblages)

Several measurement settings $x = 1, ..., m$, chosen with probability $p(x)$ (Bell non-locality, steering, QKD...):

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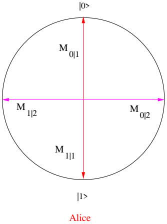
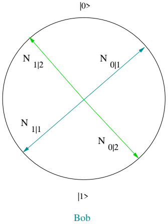
Example: Set of measurements for violation of CHSH inequality





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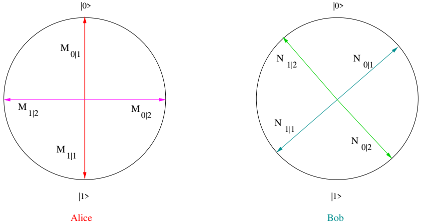


Questions:

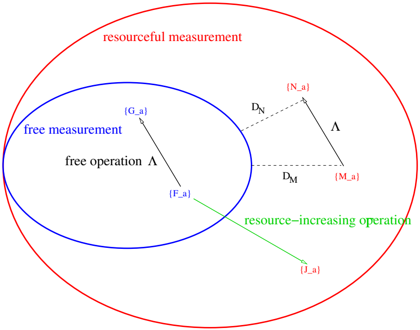
1) How to quantify resources for sets of measurements?

2) Relations between different resources of given set of measurements?

Resource theories for quantum measurements

P. Skrzypczyk and N. Linden, Phys. Rev. Lett. 122, 140403 (2019);
M. Oszmaniec and T. Biswas, Quantum 3, 133 (2019);

T. Guff et al, J. Phys. A: Math. Theor. 54, 225301 (2021)



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Resource theories for sets of quantum measurements: ingredients

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e.g. for $F_{a|x} = \sum_i \alpha_{i|(a,x)} |i\rangle\langle i|$ with $\alpha_{i|(a,x)} \geq 0 \hookrightarrow$ RT of Coherence

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2) Operations: (Free operations do not increase the resource)
(i) remember $\Lambda(\varrho) = \sum_\mu K_\mu \varrho K_\mu^\dagger$; adjoint map $\Lambda^\dagger$ acting on $M_{a|x}$:
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$\mathrm{tr}[\Lambda^\dagger(M_{a|x})\varrho]$
(ii) Mixture and classical postprocessing map:
$\mathcal{M}' = \xi(\mathcal{M})$ with $M'_{b|y} = \sum_x p(x|y) \sum_a q(b|y,x,a) M_{a|x}$

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3) Quantification of the resource: monotone $R(\mathcal{M})$ with properties
(i) $R(\mathcal{M}) = 0 \iff \mathcal{M} \in \mathcal{F}$
(ii) $R(\mathcal{M}) \geq R(\Lambda^\dagger(\mathcal{M})) \quad \forall\ \Lambda^\dagger_{free}$
(iii) $R(\mathcal{M}) \geq R(\xi(\mathcal{M})) \quad \forall\ \xi_{free}$
(iv) $R(\eta\mathcal{M} + (1-\eta)\mathcal{N}) \leq \eta R(\mathcal{M}) + (1-\eta) R(\mathcal{N})$

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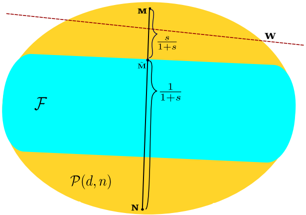
Task:

Develop a unified framework for resource theories of measurements

Resource theories for quantum measurements: quantification

Typical quantifyer in literature: robustness

$$\mathcal{R}_{\mathsf{rob}}(\mathcal{M}) := \min\left\{ s \geq 0 | \; \exists \, \mathcal{N} \, \text{s.t.} \, \frac{\mathcal{M}+s\mathcal{N}}{1+s} \in \mathcal{F} \right\}$$



M. Oszmaniec and T Biswas, Quantum 3, 133 (2019)

Note: Robustness may be not "well-behaved"
(e.g. may be infinite)

Distance-based measurement resource quantification

Introduce resource monotone:

$$R(\mathcal{M}) := \min_{\mathcal{F}\in\mathbb{F}} D(\mathcal{M},\mathcal{F})$$

where $\mathbb{F}$ is set of free measurements, $D(\mathcal{M},\mathcal{F})$ is distance (i.e. positive for $\mathcal{M} \neq \mathcal{F}$, symmetric, fulfils triangle inequality)

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Our suggestion:

Measure-and-prepare channel associated to POVM:

$$\Lambda_{\mathcal{M}_x}(\varrho) = \sum_a \mathrm{tr}[M_{a|x}\varrho]\,|a\rangle\langle a|$$

Diamond distance between channels:

$$D_\diamond(\Lambda_1,\Lambda_2) = \max_{\varrho\in S(\mathcal{H}\otimes\mathcal{H})} \frac{1}{2}||((\Lambda_1-\Lambda_2)\otimes \mathbb{1}_d)\varrho||_1$$

with trace norm $||X||_1 = \mathrm{tr}[\sqrt{X^\dagger X}\,]$

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Distance-based monotones for measurement resources

Informativeness:

$$\mathsf{IF}_\diamond(\mathcal{M}_{\mathbf{p}}) = \min_{\mathcal{F} \in \mathbb{F}_{\mathsf{UI}}} \sum_x p(x) D_\diamond(\Lambda_{\mathcal{M}_x}, \Lambda_{\mathcal{F}_x})$$

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Steerability:

$$\mathsf{S}(\vec{\sigma}_\mathbf{p}) = \frac{1}{2} \min_{\vec{\tau}\in \mathsf{LHS}} \sum_{a,x} p(x) ||\sigma_{a|x} - \tau_{a|x}||_1$$

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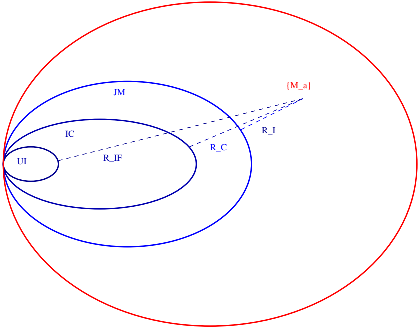
Nonlocality:

$$\mathsf{N}(\mathbf{q}_{\mathbf{p}}) = \frac{1}{2} \min_{t\in\mathsf{LHV}} \sum_{a,b,x,y} p(x,y) |q(a,b|x,y) - t(a,b|x,y)|$$

Nested structure for some types of measurement resources

UI $\equiv$ uninformative
IC $\equiv$ incoherent

JM $\equiv$ jointly measurable



Informativeness $\geq$ Coherence $\geq$ Incompatibility

Hierarchy of measurement resources

Let $\mathcal{M}_{\mathbf{p}_A}, \mathcal{N}_{\mathbf{p}_B}$ be weighted measurement assemblages and ϱ a bipartite quantum state. Let $\vec{\sigma}_{\mathbf{p}_A}$ be a state assemblage obtained via $\sigma_{a|x} = \mathrm{tr}_1[(M_{a|x} \otimes \mathbb{1})\varrho]$ and let $\mathbf{q}_\mathbf{p}$ be a probability distribution obtained via $q(a,b|x,y) = \mathrm{tr}[N_{b|y}\sigma_{a|x}]$ and $p(x,y) = p_A(x)p_B(y)$. The following sequence of inequalities holds:

$$\mathsf{IF}_\diamond(\mathcal{M}_{\mathbf{p}_A}) \geq \mathsf{C}_\diamond(\mathcal{M}_{\mathbf{p}_A}) \geq \mathsf{I}_\diamond(\mathcal{M}_{\mathbf{p}_A}) \geq \mathsf{S}(\vec{\sigma}_{\mathbf{p}_A}) \geq \mathsf{N}(\mathbf{q}_\mathbf{p})$$

Idea of proof:

i) First 2 inequalities: consequence of nested structure of free sets
ii) Third inequality: use state assemblage resulting from closest JM measurements (w.r.t. $\mathcal{M}$) $\hookrightarrow$ upper bound on steerability
iii) Fourth inequality: use probability distribution resulting from measurement on closest LHS assemblage $\hookrightarrow$ upper bound on nonlocality

L. Tendick, M. Kliesch, H. Kampermann, and DB, arXiv:2205.08546

Example for hierarchy: CGLMP measurements

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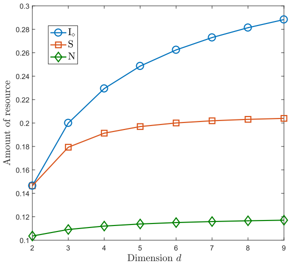

Here $\mathsf{IF}_\diamond = \mathsf{C}_\diamond = 1 - \frac{1}{d}$

Example for hierarchy: CGLMP measurements

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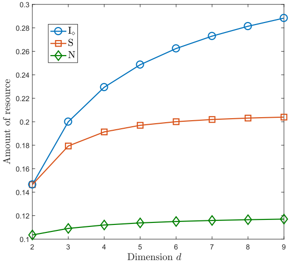

Here $\mathsf{IF}_\diamond = \mathsf{C}_\diamond = 1 - \frac{1}{d}$

Max. entangled state $\varrho = |\Phi^+\rangle\langle\Phi^+|$ with $|\Phi^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle$.
CGLMP-measurements $\{M_{a|x} = |a_x\rangle\langle a_x|\}, \{N_{b|y} = |b_y\rangle\langle b_y|\}$ in dimension d, where

$$|a_x\rangle = \frac{1}{\sqrt{d}}\sum_{q=0}^{d-1}\exp\left[\frac{2\pi i}{d}q(a-\alpha_x)\right]|q\rangle, \qquad |b_y\rangle = \frac{1}{\sqrt{d}}\sum_{q=0}^{d-1}\exp\left[-\frac{2\pi i}{d}q(b-\beta_y)\right]|q\rangle$$

with $\alpha_x = (x-1/2)/2$, $\beta_y = y/2$, and $a, b = 0, \cdots, d-1$ for $x, y = 1, 2$.

D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, Phys. Rev. Lett. 88, 040404 (2002)

Results for resource monotones

Semidefinite Programme (SDP) for diamond distance:

compare *J. Watrous, arXiv:0901.4709*

- Find numerical solutions for $R_\diamond(\mathcal{M}_\mathbf{p})$

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  - MUB measurement assemblage:
    Given $\mathcal{M} = \{M_{a|x} = |v_{a|x}\rangle\langle v_{a|x}|\}$ with $|\langle v_{a|x}|v_{b|y}\rangle| = \frac{1}{\sqrt{d}}\,\forall a, b$ and $x \neq y$. Then

$$1 - \frac{1}{m}\Big(1 + \frac{(m-1)}{\sqrt{d}}\Big) \leq I_\diamond(\mathcal{M}) \leq \frac{(d-1)(m-1)}{(d+1)m}$$

    Here $d \equiv$ dimension, $m \equiv$ number of bases.

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    Here $d \equiv$ dimension, $m \equiv$ number of bases.
  - For fixed m, large d: $\quad I_\diamond(\mathcal{M}) \approx 1 - \dfrac{1}{m}$
  - Method provides bounds for incompatibility even for cases when unknown whether MUB exists, e.g. $m = 4$ in $d = 6$.

Tightness of hierarchy

$$\mathsf{IF}_\diamond(\mathcal{M}_{\mathbf{p}_A}) \geq \mathsf{C}_\diamond(\mathcal{M}_{\mathbf{p}_A}) \geq \mathsf{I}_\diamond(\mathcal{M}_{\mathbf{p}_A}) \geq \mathsf{S}(\vec{\sigma}_{\mathbf{p}_A}) \geq \mathsf{N}(\mathbf{q}_{\mathbf{p}})$$

Question: For which measurements (and states) are the bounds (which ones?) tight?

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1) The equality

$$\mathsf{IF}_\diamond(\mathcal{M}_{\mathbf{p}}) = \mathsf{C}_\diamond(\mathcal{M}_{\mathbf{p}})$$

holds for $\mathcal{M}_{\mathbf{p}}$ which are mutually unbiased to set of projective measurements onto incoherent basis, i.e. to $\{|i\rangle\langle i|\}$.

Proof idea:
i) Show that $\mathsf{IF}_\diamond(\mathcal{M}_{\mathbf{p}}) = 1 - \frac{1}{d}$ for all rank-1 projective measurements.
ii) Show that $\mathsf{C}_\diamond(\mathcal{M}_{\mathbf{p}}) = 1 - \frac{1}{d}$ is achieved by mutually unbiased measurement.

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2) The equality

$$\mathsf{I}_\diamond(\mathcal{M}) = \mathsf{S}(\vec{\sigma})$$

holds for state $\varrho = |\Phi^+\rangle\langle\Phi^+|$ with $|\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle$ and uniformly weighted MUB measurement assemblages with $m = 2, m = d$ and $m = d + 1$.

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Proof idea:

i) Use standard construction of MUBs (finite fields with d elements), and link to depolarising robustness of measurement assemblage
$\hookrightarrow$ analytic expression for $\mathsf{I}_\diamond(\mathcal{M})$.

ii) Show via steering inequality violation that $\mathsf{S}(\vec{\sigma})$ is lower bounded by same expression.

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Conjecture: Equivalence of incompatibility and steering holds also for other constructions of MUBs and $2 \leq m \leq d + 1$.

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3) Remaining two inequalities:

We conjecture that

$$\mathsf{C}_\diamond(\mathcal{M}_{\mathbf{p}}) \geq \mathsf{I}_\diamond(\mathcal{M}_{\mathbf{p}})$$

and

$$\mathsf{S}(\vec{\sigma}_{\mathbf{p}_A}) \geq \mathsf{N}(\mathbf{q}_{\mathbf{p}})$$

are true inequalities in non-trivial scenarios (suggested by numerical search)
$\hookrightarrow$ future research needed.

Summary and Outlook

- Resource theories for quantum measurements are being developped.
- Suggestion for unified framework for resource theories of sets of measurements (quantifier: diamond distance).
- Hierarchy for measurement resources:
  Informativeness $\geq$ Coherence $\geq$ Incompatibility $\geq$ Steerability $\geq$ Nonlocality
- SDP formulation leads to numerical solutions and analytical bounds/solutions/insights (e.g. for MUB measurement assemblages).

- Future directions:
  Relations to other resource theories (e.g. imaginarity)?
  Comparison with entropic resource quantifiers for sets of measurements? Tightness of parts of hierarchy?

L. Tendick, M. Kliesch, H. Kampermann, and DB, arXiv:2205.08546

Quantum Information Theory in Düsseldorf

Institut für Theoretische Physik III, Heinrich-Heine-Universität Düsseldorf, Germany

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