Divisibility and information flow for non-invertible qubit dynamical maps

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joint work with Á. Rivas, and E. Størmer, and S. Chakraborty


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Dynamical map

$$\text{Quantum evolution} \;\longleftrightarrow\; \rho \to \rho_t := \Lambda_t(\rho)$$

$$\Lambda_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \; ; \; t \geq 0$$

- completely positive
- trace-preserving
- $\Lambda_{t=0} = \mathrm{id}$

Markovian semigroup

$$\partial_t \Lambda_t = \mathcal{L} \Lambda_t$$

Theorem (Gorini-Kossakowski-Sudarshan-Lindblad (1976))

$\Lambda_t = e^{t\mathcal{L}}$ *is CPTP if and only if*

$$\mathcal{L}(\rho) = -i[H,\rho] + \sum_\alpha \gamma_\alpha \left(V_\alpha \rho V_\alpha^\dagger - \frac{1}{2} \left(V_\alpha^\dagger V_\alpha \rho + \rho V_\alpha^\dagger V_\alpha \right) \right) \; ; \; \gamma_\alpha > 0$$

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A Brief History of the GKLS Equation

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Abstract. We reconstruct the chain of events, intuitions and ideas that led to the formulation of the Gorini, Kossakowski, Lindblad and Sudarshan equation.

Beyond Markovian semigroup

- non-local master equation (Nakajima-Zwanzig equation)

$$\partial_t \Lambda_t = \int_0^t K_{t-\tau} \Lambda_\tau \, d\tau$$

  - semi-Markov (Breuer-Vacchini, Kossakowski, DC)
  - collision models (Giovanetti, Palma, Cicarello, Lorenzo ...)
  - Engineering quantum channels (Marshall, Zanardi, ...)

- local in time master equation (TCL)

$$\partial_t \Lambda_t = \mathcal{L}_t \Lambda_t$$

Invertible maps

$$\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \quad \text{(CPTP)}$$

$$\Phi^{-1} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$$

Φ^{-1} is trace-preserving but not CP

Theorem

Φ^{-1} is CPTP iff

$$\Phi(\rho) = U \rho U^\dagger \quad \longrightarrow \quad \Phi^{-1}(X) = U^\dagger X U$$

Invertible maps

$$\partial_t \Lambda_t = \mathcal{L}_t \Lambda_t \ , \quad \Lambda_0 = \mathrm{id}$$

$$\mathcal{L}_t = [\partial_t \Lambda_t] \, \Lambda_t^{-1}$$

$\mathcal{L}_t$ — regular generator

Example: qubit map

$$\Lambda_t(\rho) = \begin{pmatrix} \rho_{11} & \rho_{12}\, f(t) \\ \rho_{21}\, f^*(t) & \rho_{22} \end{pmatrix} \;\; ; \quad f(t) \in \mathbb{C}$$

Λ_t is CPTP iff $|f(t)| \leq 1$

Λ_t is invertible iff $f(t) \neq 0$

$$\boxed{\mathcal{L}_t(\rho) = -i\omega(t)[\sigma_3, \rho] + \gamma(t)[\sigma_3 \rho \sigma_3 - \rho]}$$

$$\gamma(t) = -\mathrm{Re}\,\frac{\dot{f}(t)}{f(t)} \;\; ; \quad \omega(t) = -\mathrm{Im}\,\frac{\dot{f}(t)}{f(t)}$$

Divisibility vs Markovianity

Divisibility

$$\Lambda_t = V_{t,s}\,\Lambda_s \quad ; \; t \geq s$$

$$V_{t,s} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$$

- P-divisible iff $V_{t,s}$ is PTP
- CP-divisible iff $V_{t,s}$ is CPTP

CP-divisible $\implies$ P-divisible

Theorem (Benatti,DC,Fillipov (2017))

Λ_t is CP-divisible iff $\Lambda_t \otimes \Lambda_t$ is P-divisible

Invertible maps

Invertible map is always divisible

$$V_{t,s} := \Lambda_t \, \Lambda_s^{-1}$$

Theorem

If Λ_t is **invertible**, *then it is CP-divisible iff*

$$L_t(\rho) \;=\; -i[H(t),\rho]+\sum_\alpha \gamma_\alpha(t)\left([V_\alpha(t),\rho V_\alpha^\dagger(t)] + [V_\alpha(t)\rho, V_\alpha^\dagger(t)]\right)$$

and $\gamma_\alpha(t) \geq 0$.

Markovianity - RHP

Definition: [Rivas,Huelga,Plenio]

Evolution is MARKOVIAN iff Λ_t is CP-divisible

Markovianity - BLP

Definition: [Breuer,Laine,Piilo]

MARKOVIANITY $\longleftrightarrow$ no backflow of information

$$\frac{d}{dt}\|\Lambda_t(\sigma - \rho)\|_1 \leq 0$$

$$\|X\|_1 = \mathrm{Tr}\sqrt{XX^\dagger}$$

Markovian vs. non-Markovian

- Markovianity is defined for classical stochastic processes
- Quantum Markovian processes (Accardi, Frigero, Lewis, Lindblad)
- Markovianity = CP-divisibility (Rivas, Huelga, Plenio)
- Markovianity = no information flow (Breuer, Laine, Piilo)
- Geometrical characterization of non-Markovianity (Lorenzo, Plastina, Paternostro)
- non-Markovianity via mutual information (Luo)
- non-Markovianity via quantum regression theorem (Vacchini, Smirne, Huelga, Petruccione,...)
- non-Markovianity via discrimination of states (Buscemi, Datta)
- non-Markovianity via channel capacity (Bylicka, DC, Maniscalco)
- ...

Divisibility vs. monotonicity of trace norm

Theorem

If Λ_t is P-divisible, then

$$\frac{d}{dt} \| \Lambda_t(X) \|_1 \leq 0$$

for any $X \in \mathcal{B}(\mathcal{H})$.

$$X = \rho - \sigma$$

CP-divisibility $\Longrightarrow$ P-divisibility $\Longrightarrow$ no information backflow

$$\{p_1, \rho_1; p_2, \rho_2\} \longrightarrow D(\rho_1, \rho_2) = \| \, p_1\rho_1 - p_2\rho_2 \, \|_1$$

Theorem (DC,Kossakowski,Rivas)

If Λ_t is **invertible**, *then it is P-divisible* **if and only if**

$$\frac{d}{dt} \| \, \Lambda_t(p_1\rho_1 - p_2\rho_2) \, \|_1 \leq 0$$

for all pair ρ_1, ρ_2 and p_1, p_2.
It is CP-divisible **if and only if**

$$\frac{d}{dt} \| \, \mathrm{id}_d \otimes \Lambda_t(p_1\widetilde{\rho}_1 - p_2\widetilde{\rho}_2) \, \|_1 \leq 0$$

for all pair $\widetilde{\rho}_1, \widetilde{\rho}_2$ and p_1, p_2.

Theorem (Bylicka,Johansson,Acín)

If Λ_t *is* **invertible**, *then it is CP-divisible* **if and only if**

$$\frac{d}{dt}\| \, \mathrm{id}_{d+1} \otimes \Lambda_t(\widetilde{\rho}_1 - \widetilde{\rho}_2) \, \|_1 \leq 0$$

for all pair $\widetilde{\rho}_1, \widetilde{\rho}_2$.

Example: qubit evolution

$$\mathcal{L}_t(\rho) = \frac{1}{2}\sum_{k=1}^{3}\gamma_k(t)[\sigma_k\rho\sigma_k - \rho] =: \sum_k \gamma_k(t)\mathcal{L}_k$$

Λ_t is invertible $\iff \Gamma_k(t) = \int_0^t \gamma_k(\tau)d\tau < \infty$

- Λ_t is CP-divisible iff

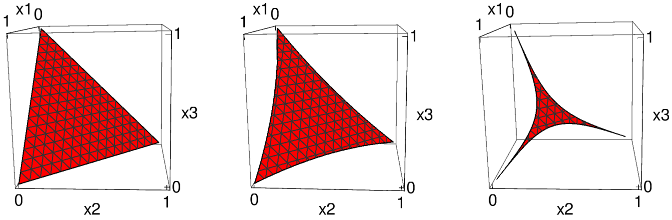
$$\gamma_1(t) \geq 0 \;;\quad \gamma_2(t) \geq 0 \;;\quad \gamma_3(t) \geq 0$$

- Λ_t is P-divisible iff

$$\gamma_1(t) + \gamma_2(t) \geq 0 \;;\quad \gamma_1(t) + \gamma_3(t) \geq 0 \;;\quad \gamma_2(t) + \gamma_3(t) \geq 0$$

$\Lambda_t = x_1 e^{t\mathcal{L}_1} + x_2 e^{t\mathcal{L}_2} + x_3 e^{t\mathcal{L}_3}$

- Λ_t is a Markovian semi-group if only one $x_k = 1$
- Λ_t is P-divisible for all x_k
- Λ_t is CP-divisible for "small region"

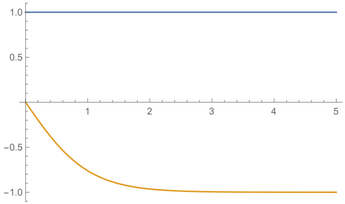


N.Megier, DC, J.Piilo, W.Strunz, Sc. Rep. 2017

$\Lambda_t = \frac{1}{2}\left(e^{t\mathcal{L}_1} + e^{t\mathcal{L}_2}\right)$

$$\mathcal{L}_t(\rho) = \frac{1}{2}\sum_{k=1}^{3} \gamma_k(t)[\sigma_k \rho \sigma_k - \rho]$$

$\gamma_1(t) = \gamma_2(t) = 1$, $\gamma_3(t) = -\tanh t$ ‘eternally non-Markovian’



Hall, Cresser, Andersson, PRA 2014

But what if the map is not invertible?

- F. Buscemi and N. Datta, Phys. Rev. A (2016)
- DC, A. Rivas, and E. Størmer, Phys. Rev. Lett. (2018).
- S. Chakraborty, DC, Phys. Rev A (2019)

General Λ_t

$$\Lambda_t = V_{t,s}\,\Lambda_s \quad ; \; t \geq s$$

Theorem

Λ_t is divisible iff it is "kernel non-decreasing"

$$\mathrm{Ker}\Lambda_s \subset \mathrm{Ker}\Lambda_t$$

$V_{t,s}$ is NOT uniquely defined on $\mathcal{B}(\mathcal{H})$

$$V_{t,s} : \mathrm{Im}\,\Lambda_s \to \mathrm{Im}\,\Lambda_t$$

$$\mathrm{Extension}: \;\; \longrightarrow \;\; \widetilde{V}_{t,s} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$$

General Λ_t

Theorem

$$\text{If} \quad \frac{d}{dt} \| \, [\mathbb{1} \otimes \Lambda_t](p_1 \rho_1 - p_2 \rho_2) \, \|_1 \leq 0$$

for all $\rho_1, \rho_2 \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$, *and* p_1, p_2, *then*

1. Λ_t *is divisible*
2. $V_{t,s}$ *is CPTP on* $\mathrm{Im}\, \Lambda_s$.

Could we extend $V_{t,s}$ to $\mathcal{B}(\mathcal{H})$?

Arveson extension theorem

$$M \subset \mathcal{B}(\mathcal{H}) \quad \text{– operator system}$$

- $\mathbb{I} \in M$
- $X \in M \implies X^\dagger \in M$

Theorem (Arveson)

Let M be an operator system and $\Phi : M \to \mathcal{B}(\mathcal{H})$ a unital CP map. Then there exists unital CP extension $\widetilde{\Phi} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$.

Generalizing Arveson theorem

$M \subset \mathcal{B}(\mathcal{H})$ – spanned by positive operators

Theorem (Jencova)

If $\Phi : M \to \mathcal{B}(\mathcal{H})$ *is a CP map, then there exists CP extension* $\widetilde{\Phi} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$.

Theorem (DC,Rivas,Størmer)

$$\text{If} \quad \frac{d}{dt} \| \, [\mathbb{1} \otimes \Lambda_t](p_1\rho_1 - p_2\rho_2) \, \|_1 \leq 0$$

for all $\rho_1, \rho_2 \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$*, and* p_1, p_2*, then* $V_{t,s}$ *is CPTP on* $\mathrm{Im}\,\Lambda_s$*, and it can be extended to CP map*

$$\widetilde{V}_{t,s} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$$

$\widetilde{V}_{t,s}$ is always trace-preserving on $\mathrm{Im}\,\Lambda_s$

$\widetilde{V}_{t,s}$ needs NOT be trace-preserving !!!

Could we have both CP and trace-preservation ?

$$\Lambda_t = V_{t,s}\,\Lambda_s \quad ; \; t \geq s$$

Λ_t is "image non-increasing" $\iff \mathrm{Im}\Lambda_t \subset \mathrm{Im}\Lambda_s$

Theorem (DC,Rivas,Størmer)

If Λ_t is image non-increasing, then it is CP-divisible **if and only if**

$$\frac{d}{dt}\,\|\,[\mathbb{1}\otimes\Lambda_t](p_1\rho_1 - p_2\rho_2)\,\|_1 \leq 0$$

for all $\rho_1, \rho_2 \in \mathcal{B}(\mathcal{H}\otimes\mathcal{H})$, and p_1, p_2,

Example – image non-increasing

- $\Lambda_{t_1}\Lambda_{t_2} = \Lambda_{t_2}\Lambda_{t_1}$
- Λ_t is diagonalizable

Theorem

If Λ_t is kernel non-decreasing, then it is image non-increasing.

A lot of studied examples fit this class

Qubit

Theorem (DC, S. Chakraborty)

Λ_t *is CP-divisible* **if and only if**

$$\frac{d}{dt} \| [\mathbb{1} \otimes \Lambda_t](p_1\rho_1 - p_2\rho_2) \|_1 \leq 0$$

for all ρ_1, ρ_2 *and* p_1, p_2.

Qubit channels

$$\mathbf{B} = \text{Bloch ball} \quad ; \quad \mathbf{S} = \text{Bloch sphere}$$

$$\Phi(\mathbf{B}) \subset \mathbf{B}$$

$$\boxed{\text{Pure Output} \quad \longrightarrow \quad \mathbf{PO} = \Phi(\mathbf{B}) \cap \mathbf{S}}$$

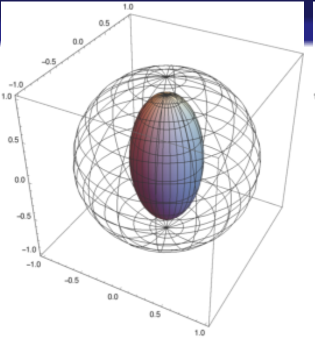
Qubit channels

Pure Output $\longrightarrow$ $\mathbf{PO} = \Phi(\mathbf{B}) \cap \mathbf{S}$

Theorem (Braun, Giraud, Nechita, Pellegrini, Žnidarič (2014))

For any qubit channel

- $\mathbf{PO} = \{\emptyset\}$
- $|\mathbf{PO}| = 1$
- $|\mathbf{PO}| = 2$
- $|\mathbf{PO}| > 2 \implies \mathbf{PO} = \mathbf{S}$; $(\Phi(\rho) = U\rho U^\dagger)$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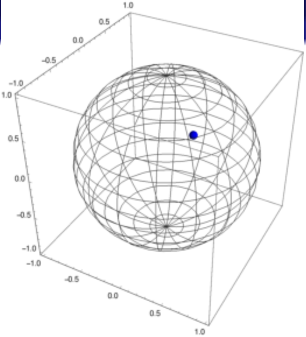
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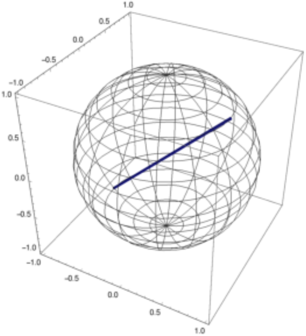
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0.5
1.0

Qubit channels

Proposition (Chakraborty, DC (2019))

There is NO qubit CPTP projector such that $\dim \mathrm{Im}(\Phi) = 3$.

Quantum channel with 3-dim. image

Example

$$\Phi(\rho) = \frac{1}{2}\rho + \frac{1}{4}\Big(\sigma_1\rho\sigma_1 + \sigma_2\rho\sigma_2\Big).$$

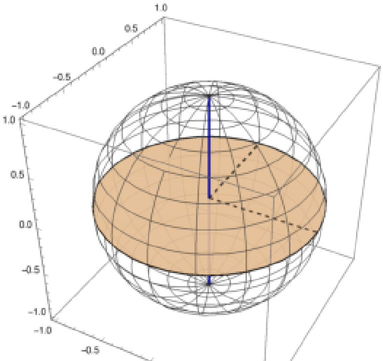
$$\Phi(\mathbb{1}) = \mathbb{1},\ \Phi(\sigma_1) = \frac{1}{2}\sigma_1,\ \Phi(\sigma_2) = \frac{1}{2}\sigma_2,\ \Phi(\sigma_3) = 0,$$

- $\dim \mathrm{Im}(\Phi) = 3.$
- Φ *is not a projector!*

CPTP vs. PTP

$$\mathcal{S}_z(\rho) = \frac{1}{2}\Big(\rho + \sigma_3\rho\sigma_3\Big) \quad \text{CPTP projector}$$

$$\Pi_{xy}(\rho) = \frac{1}{4}\Big(3\rho + \sigma_1\rho\sigma_1 + \sigma_2\rho\sigma_2 - \sigma_3\rho\sigma_3\Big) \quad \text{PTP projector}$$



If the image is 2 dim.

$$\{\rho_1, \rho_2\} \quad ; \quad \{\sigma_1, \sigma_2\}$$

Does there exist a quantum chanel Φ such that $\sigma_k = \Phi(\rho_k)$?

Theorem (Alberti and Uhlmann)

$$\| \rho_1 - t\rho_2\|_1 \geq \| \sigma_1 - t\sigma_2\|_1$$

for all $t > 0$. *Equivalently*

$$\| p_1\rho_1 - p_2\rho_2\|_1 \geq \| p_1\sigma_1 - p_2\sigma_2\|_1$$

for all $p_1 + p_2 = 1$.

The theorem is NOT true for higher dimensions!

Conclusions

- Markovianity = CP-divisiblity $\longleftrightarrow$ $\Lambda_t = V_{t,s} \circ \Lambda_s$
- for invertible maps: CP-divisibility $\longleftrightarrow$ monotonicity of the trace norm
- we generalize it for non-invertible maps **but**
- for non-invertible maps: $V_{t,s}$ might be CP on $\mathcal{B}(\mathcal{H})$ but trace-preserving only on $\mathrm{Im}\Lambda_s$
- for image decreasing maps it is also trace-preserving on $\mathcal{B}(\mathcal{H})$
- for the qubit evolution: CP-divisibility $\longleftrightarrow$ monotonicity of the trace norm