Quantifying the incompatibility of quantum measurements relative to a basis

Georgios Styliaris

Paolo Zanardi

arXiv: 1901.06382

USC University of Southern California

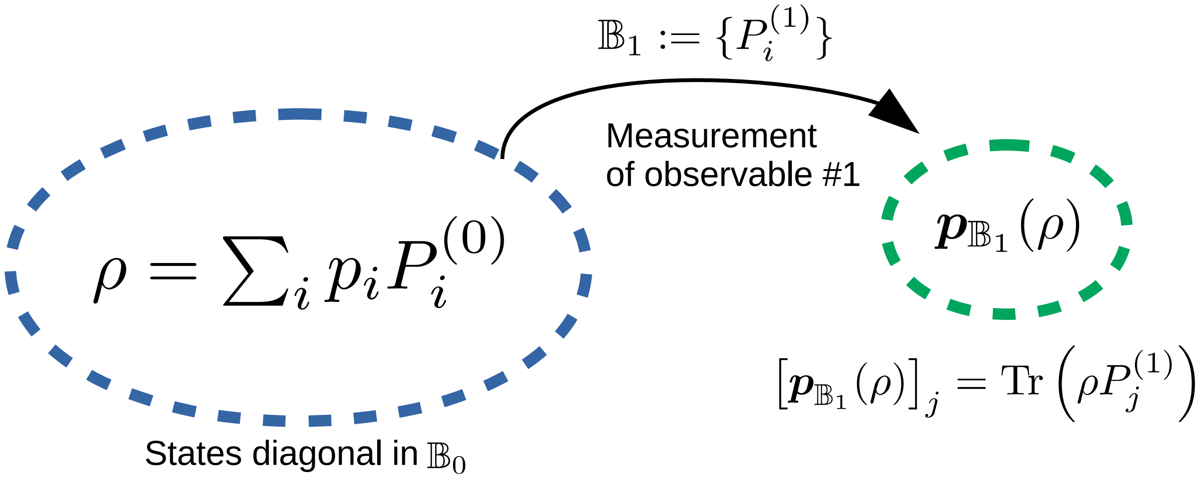
Incompatibility relative to $\mathbb{B}_0$: Orthogonal Measurements

$$\rho = \sum_i p_i P_i^{(0)}$$

States diagonal in $\mathbb{B}_0$

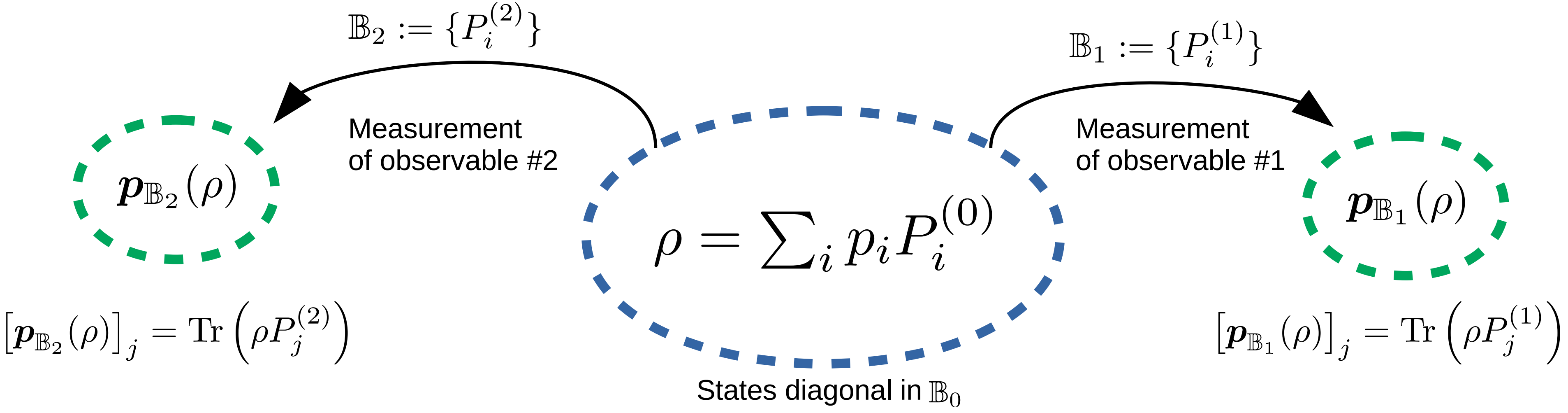
"Basis" $\mathbb{B} := \{P_i\}_{i=1}^{d}$, where $P_i := |i\rangle\langle i|$

Incompatibility relative to $\mathbb{B}_0$: Orthogonal Measurements



"Basis" $\mathbb{B} := \{P_i\}_{i=1}^{d}$, where $P_i := |i\rangle\langle i|$

Incompatibility relative to $\mathbb{B}_0$: Orthogonal Measurements



"Basis" $\mathbb{B} := \{P_i\}_{i=1}^{d}$, where $P_i := |i\rangle\langle i|$

Incompatibility relative to $\mathbb{B}_0$: Orthogonal Measurements

$\mathbb{B}_2 := \{P_i^{(2)}\}$

$\mathbb{B}_1 := \{P_i^{(1)}\}$

Measurement of observable #2

Measurement of observable #1

$\boldsymbol{p}_{\mathbb{B}_2}(\rho)$

$$\rho = \sum_i p_i P_i^{(0)}$$

$\boldsymbol{p}_{\mathbb{B}_1}(\rho)$

$$\left[\boldsymbol{p}_{\mathbb{B}_2}(\rho)\right]_j = \mathrm{Tr}\left(\rho P_j^{(2)}\right)$$

$$\left[\boldsymbol{p}_{\mathbb{B}_1}(\rho)\right]_j = \mathrm{Tr}\left(\rho P_j^{(1)}\right)$$

States diagonal in $\mathbb{B}_0$

"Basis" $\mathbb{B} := \{P_i\}_{i=1}^{d}$, where $P_i := |i\rangle\langle i|$

$$[X(\mathbb{B}_k, \mathbb{B}_0)]_{ij} := \mathrm{Tr}\left(P_i^{(k)} P_j^{(0)}\right)$$

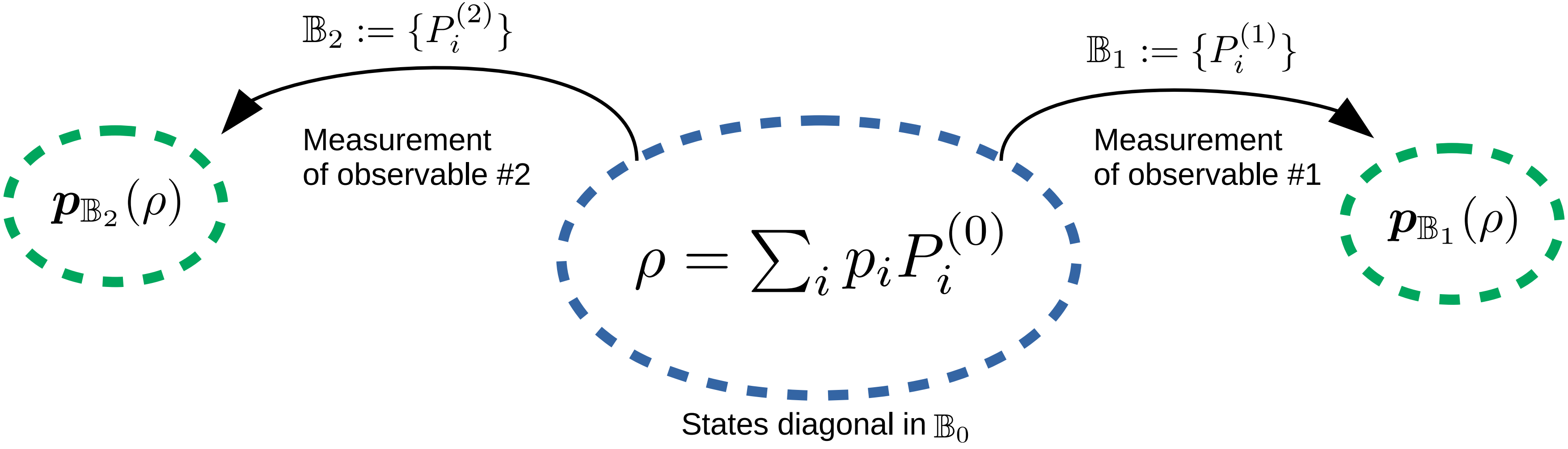
$$\boldsymbol{p}_{\mathbb{B}_k}(\rho) = X(\mathbb{B}_k, \mathbb{B}_0)\boldsymbol{p}$$

$(k = 1, 2)$

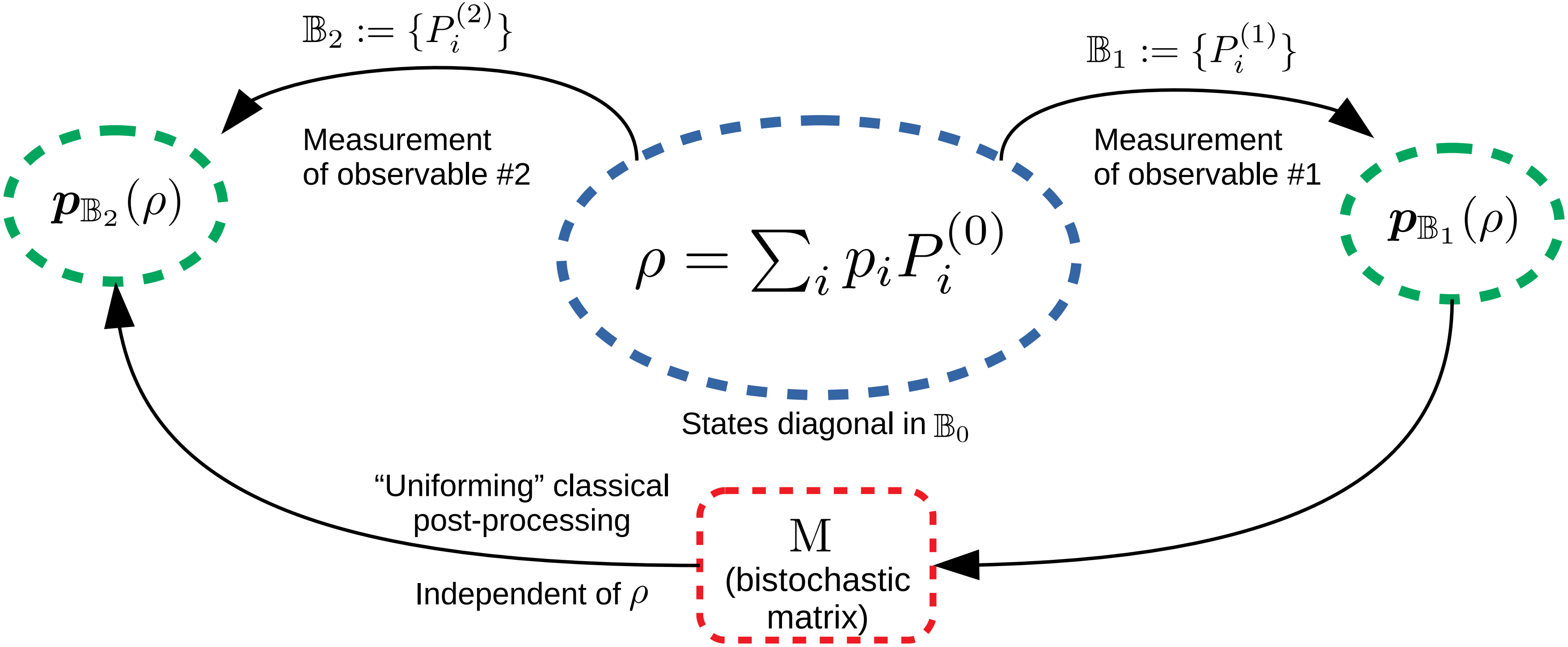
"transition matrix" between two bases is **bistochastic**

Output probability distributiton is **more uniform** than input one

Incompatibility relative to $\mathbb{B}_0$: Orthogonal Measurements



Incompatibility relative to $\mathbb{B}_0$: Orthogonal Measurements



$$\boldsymbol{p}_{\mathbb{B}_2}(\rho) = M\boldsymbol{p}_{\mathbb{B}_1}(\rho) \quad \forall \rho \text{ diagonal in } \mathbb{B}_0 \quad \Longleftrightarrow \quad \boxed{X(\mathbb{B}_2, \mathbb{B}_0) = MX(\mathbb{B}_1, \mathbb{B}_0)}$$

Incompatibility relative to $\mathbb{B}_0$: Orthogonal Measurements

Notation: we write $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$ if and only if there exists a bistochastic matrix M such that $X(\mathbb{B}_2, \mathbb{B}_0) = MX(\mathbb{B}_1, \mathbb{B}_0)$

"An orthogonal measurement over $\mathbb{B}_1$ is more compatible than over $\mathbb{B}_2$, relative to states diagonal in $\mathbb{B}_0$"

Incompatibility relative to $\mathbb{B}_0$: Orthogonal Measurements

Notation: we write $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$ if and only if there exists a bistochastic matrix M such that $X(\mathbb{B}_2, \mathbb{B}_0) = MX(\mathbb{B}_1, \mathbb{B}_0)$

"An orthogonal measurement over $\mathbb{B}_1$ is more compatible than over $\mathbb{B}_2$, relative to states diagonal in $\mathbb{B}_0$"

Properties:

- "$\succ^{\mathbb{B}_0}$" is a preorder over bases, i.e.,

  i) $\mathbb{B} \succ^{\mathbb{B}_0} \mathbb{B} \ \forall \mathbb{B}$ (reflexivity) and

  ii) $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$, $\mathbb{B}_2 \succ^{\mathbb{B}_0} \mathbb{B}_3$ imply $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_3$ (transitivity)

Resource theories!

Incompatibility relative to $\mathbb{B}_0$: Orthogonal Measurements

Notation: we write $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$ if and only if there exists a bistochastic matrix M such that $X(\mathbb{B}_2, \mathbb{B}_0) = MX(\mathbb{B}_1, \mathbb{B}_0)$

"An orthogonal measurement over $\mathbb{B}_1$ is more compatible than over $\mathbb{B}_2$, relative to states diagonal in $\mathbb{B}_0$"

Properties:

- "$\succ^{\mathbb{B}_0}$" is a preorder over bases, i.e.,
  i) $\mathbb{B} \succ^{\mathbb{B}_0} \mathbb{B} \ \forall \mathbb{B}$ (reflexivity) and
  ii) $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$, $\mathbb{B}_2 \succ^{\mathbb{B}_0} \mathbb{B}_3$ imply $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_3$ (transitivity)
- $\mathbb{B}_0 \succ^{\mathbb{B}_0} \mathbb{B} \ \forall \mathbb{B}$

Resource theories!

"Measurement over $\mathbb{B}_0$ is more compatible than over any other basis"

Incompatibility relative to $\mathbb{B}_0$: Orthogonal Measurements

Notation: we write $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$ if and only if there exists a bistochastic matrix M such that $X(\mathbb{B}_2, \mathbb{B}_0) = MX(\mathbb{B}_1, \mathbb{B}_0)$

"An orthogonal measurement over $\mathbb{B}_1$ is more compatible than over $\mathbb{B}_2$, relative to states diagonal in $\mathbb{B}_0$"

Properties:

- "$\succ^{\mathbb{B}_0}$" is a preorder over bases, i.e.,
  i) $\mathbb{B} \succ^{\mathbb{B}_0} \mathbb{B} \;\forall \mathbb{B}$ (reflexivity) and
  ii) $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$, $\mathbb{B}_2 \succ^{\mathbb{B}_0} \mathbb{B}_3$ imply $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_3$ (transitivity)

Resource theories!

- $\mathbb{B}_0 \succ^{\mathbb{B}_0} \mathbb{B} \;\forall \mathbb{B}$

"Measurement over $\mathbb{B}_0$ is more compatible than over any other basis"

- $\mathbb{B} \succ^{\mathbb{B}_0} \mathbb{B}_{\mathrm{MU}} \;\forall \mathbb{B}$, where $\mathbb{B}_{\mathrm{MU}}$ is any basis mutually unbiased to $\mathbb{B}_0$

"Measurement over any basis is more compatible than over any mutually unbiased one"

The “Quantum” version of $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$

Notation: we write $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$ if and only if there exists a bistochastic matrix M such that $X(\mathbb{B}_2, \mathbb{B}_0) = M X(\mathbb{B}_1, \mathbb{B}_0)$

$$\mathcal{D}_{\mathbb{B}}(X) := \sum_i P_i X P_i$$

“Dephasing” or “**Measurement”** map

$$\mathbb{B} = \{P_i\}_{i=1}^d$$

The "Quantum" version of $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$

Notation: we write $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$ if and only if there exists a bistochastic matrix M such that $X(\mathbb{B}_2, \mathbb{B}_0) = MX(\mathbb{B}_1, \mathbb{B}_0)$

$$\mathcal{D}_{\mathbb{B}}(X) := \sum_i P_i X P_i$$

"Dephasing" or "**Measurement"** map

$$\mathbb{B} = \{P_i\}_{i=1}^d$$

Proposition. $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$ if and only if there exist a unitary superoperator $\mathcal{U}$ and a (possibly trivial) sequence of measurements $\{\mathcal{D}_{\mathbb{B}'_\alpha}\}_\alpha$ such that

$$\mathcal{D}_{\mathbb{B}_2}\mathcal{D}_{\mathbb{B}_0} = \mathcal{U}\left[\prod_\alpha \mathcal{D}_{\mathbb{B}'_\alpha}\right]\mathcal{D}_{\mathbb{B}_1}\mathcal{D}_{\mathbb{B}_0} \quad .$$

$\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$ can also be understood as orthogonal measurement emulation via other such measurements

Quantifying the incompatibility relative to $\mathbb{B}_0$

Notation: we write $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$ if and only if there exists a bistochastic matrix M such that $X(\mathbb{B}_2, \mathbb{B}_0) = MX(\mathbb{B}_1, \mathbb{B}_0)$

Preorder

$\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$

→

Monotones

(functions from "bases" to non-negative Real)

$f_{\mathbb{B}_0}(\mathbb{B}_1) \geq f_{\mathbb{B}_0}(\mathbb{B}_2)$ Compatibility measure

$f_{\mathbb{B}_0}(\mathbb{B}_1) \leq f_{\mathbb{B}_0}(\mathbb{B}_2)$ Incompatibility measure

Quantifying the incompatibility relative to $\mathbb{B}_0$

Notation: we write $\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$ if and only if there exists a bistochastic matrix M such that $X(\mathbb{B}_2, \mathbb{B}_0) = MX(\mathbb{B}_1, \mathbb{B}_0)$

Preorder

$\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$

→

Monotones

(functions from "bases" to non-negative Real)

$f_{\mathbb{B}_0}(\mathbb{B}_1) \geq f_{\mathbb{B}_0}(\mathbb{B}_2)$ Compatibility measure

$f_{\mathbb{B}_0}(\mathbb{B}_1) \leq f_{\mathbb{B}_0}(\mathbb{B}_2)$ Incompatibility measure

Proposition. Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a continuous convex (concave) function. Then,

(i) $f^{\phi}_{\mathbb{B}_0}(\mathbb{B}_1) := \sum_i \phi(X^R_i(\mathbb{B}_1, \mathbb{B}_0))$ is a measure of relative compatibility (incompatibility).

(ii) The family $\{f^{\phi}_{\mathbb{B}_0}\}_{\phi}$ for all continuous convex (concave) is a complete set of monotones.

Incompatibility and Quantum Coherence

$$c_{\mathbb{B}}(\rho) := S(\rho \| \mathcal{D}_{\mathbb{B}} \rho)$$

“Relative entropy of coherence”

Interpretation as:

i) Disturbance by a measurement

(statistical intepretation of KL divergence)

ii) Rate of distillable coherence

(Coherence as a Resource Theory)

Incompatibility and Quantum Coherence

$$c_{\mathbb{B}}(\rho) := S(\rho \| \mathcal{D}_{\mathbb{B}}\rho)$$

"Relative entropy of coherence"

$$\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$$

Incompatibility relative to $\mathbb{B}_0$

$$c_{\mathbb{B}_1}(\rho_0) \leq c_{\mathbb{B}_2}(\rho_0) \quad \forall \rho_0 \text{ diagonal in } \mathbb{B}_0$$

The **more incompatible** two bases are, the **more coherence** states have.

Interpretation as:

i) Disturbance by a measurement

(statistical intepretation of KL divergence)

ii) Rate of distillable coherence

(Coherence as a Resource Theory)

Incompatibility and Quantum Coherence

$$c_{\mathbb{B}}(\rho) := S(\rho \| \mathcal{D}_{\mathbb{B}}\rho)$$

"Relative entropy of coherence"

$\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$ Incompatibility relative to $\mathbb{B}_0$

$$c_{\mathbb{B}_1}(\rho_0) \leq c_{\mathbb{B}_2}(\rho_0) \quad \forall \rho_0 \text{ diagonal in } \mathbb{B}_0$$

The **more incompatible** two bases are, the **more coherence** states have.

Interpretation as:

i) Disturbance by a measurement

(statistical intepretation of KL divergence)

ii) Rate of distillable coherence

(Coherence as a Resource Theory)

$\langle c_{\mathbb{B}_1}(\rho_0) \rangle_{\rho_0}$ **is a measure of incompatibility!**

(uniform averaging over states diagonal in $\mathbb{B}_0$)

+ convertibility results in Resource Theories of Coherence

Incompatibility and Uncertainty Relations

Entropic Uncertainty

$$S(p_{\mathbb{B}_1})(\rho_0) + S(p_{\mathbb{B}_2})(\rho_0) \geq S(\rho_0) + r^{(\mathrm{MU})}(\mathbb{B}_2, \mathbb{B}_1)$$

$$r^{(\mathrm{MU})} := -\log(\max_{i,j} X_{ij}(\mathbb{B}_1, \mathbb{B}_2))$$

by Maasen & Uffink + Coles *et al.*

Incompatibility and Uncertainty Relations

Entropic Uncertainty

$$S(p_{\mathbb{B}_1})(\rho_0) + S(p_{\mathbb{B}_2})(\rho_0) \geq S(\rho_0) + r^{(\mathrm{MU})}(\mathbb{B}_2, \mathbb{B}_1)$$

$$r^{(\mathrm{MU})} := -\log(\max_{i,j} X_{ij}(\mathbb{B}_1, \mathbb{B}_2))$$

by Maasen & Uffink + Coles *et al.*

Quantum Fluctuations

state diagonal in $\mathbb{B}_0$ and observable diagonal in $\mathbb{B}_1$

$$Q_{\mathbb{B}_0}(\mathbb{B}_1) \coloneqq \sup_{A \in \mathcal{A}_{\mathbb{B}_1}, \|A\|_2 = 1} \max_{i=1,\dots,d} \mathrm{Var}_i(A)$$

Incompatibility and Uncertainty Relations

Entropic Uncertainty

$$S(p_{\mathbb{B}_1})(\rho_0) + S(p_{\mathbb{B}_2})(\rho_0) \geq S(\rho_0) + r^{(\mathrm{MU})}(\mathbb{B}_2, \mathbb{B}_1)$$

$$r^{(\mathrm{MU})} := -\log(\max_{i,j} X_{ij}(\mathbb{B}_1, \mathbb{B}_2))$$

by Maasen & Uffink + Coles *et al.*

Quantum Fluctuations

state diagonal in $\mathbb{B}_0$ and observable diagonal in $\mathbb{B}_1$

$$Q_{\mathbb{B}_0}(\mathbb{B}_1) := \sup_{A \in \mathcal{A}_{\mathbb{B}_1}, \|A\|_2 = 1} \max_{i=1,\dots,d} \mathrm{Var}_i(A)$$

$$\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$$

Incompatibility relative to $\mathbb{B}_0$

$$r^{(\mathrm{MU})}(\mathbb{B}_1, \mathbb{B}_0) \leq r^{(\mathrm{MU})}(\mathbb{B}_2, \mathbb{B}_0)$$

$r^{(\mathrm{MU})}(\mathbb{B}_1, \mathbb{B}_0)$ **is a measure of incompatibility! (w.r.t. first argument)**

Incompatibility and Uncertainty Relations

Entropic Uncertainty

$$S(p_{\mathbb{B}_1})(\rho_0) + S(p_{\mathbb{B}_2})(\rho_0) \geq S(\rho_0) + r^{(\mathrm{MU})}(\mathbb{B}_2, \mathbb{B}_1)$$

$$r^{(\mathrm{MU})} := -\log(\max_{i,j} X_{ij}(\mathbb{B}_1, \mathbb{B}_2))$$

by Maasen & Uffink + Coles *et al.*

Quantum Fluctuations

state diagonal in $\mathbb{B}_0$ and observable diagonal in $\mathbb{B}_1$

$$Q_{\mathbb{B}_0}(\mathbb{B}_1) := \sup_{A \in \mathcal{A}_{\mathbb{B}_1}, \|A\|_2 = 1} \max_{i=1,\ldots,d} \mathrm{Var}_i(A)$$

$$\mathbb{B}_1 \succ^{\mathbb{B}_0} \mathbb{B}_2$$

Incompatibility relative to $\mathbb{B}_0$

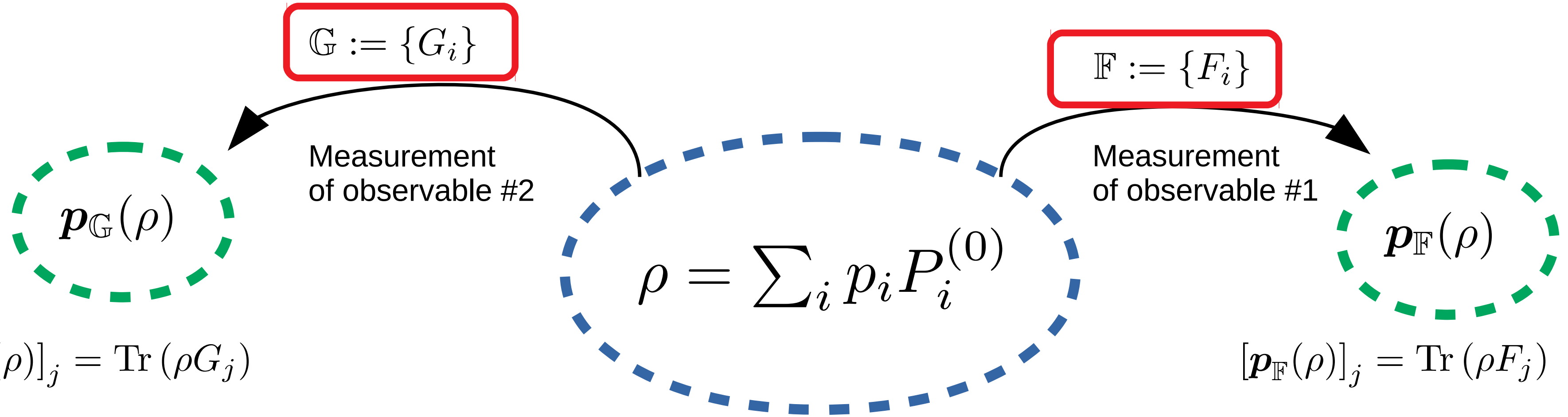
$$r^{(\mathrm{MU})}(\mathbb{B}_1, \mathbb{B}_0) \leq r^{(\mathrm{MU})}(\mathbb{B}_2, \mathbb{B}_0)$$

$r^{(\mathrm{MU})}(\mathbb{B}_1, \mathbb{B}_0)$ **is a measure of incompatibility! (w.r.t. first argument)**

$$Q_{\mathbb{B}_0}(\mathbb{B}_1) \leq 1 - \lambda_{\min}\left(X(\mathbb{B}_1, \mathbb{B}_0) X^T(\mathbb{B}_1, \mathbb{B}_0)\right) := q(\mathbb{B}_1, \mathbb{B}_0)$$

$q(\mathbb{B}_1, \mathbb{B}_0) \leq q(\mathbb{B}_2, \mathbb{B}_0)$ **is a measure of incompatibility! (w.r.t. first argument)**

Incompatibility relative to $\mathbb{B}_0$: Generalized Measurements



$$[\boldsymbol{p}_{\mathbb{G}}(\rho)]_j = \mathrm{Tr}\,(\rho G_j)$$

$$[\boldsymbol{p}_{\mathbb{F}}(\rho)]_j = \mathrm{Tr}\,(\rho F_j)$$

States diagonal in $\mathbb{B}_0$

"Basis" $\mathbb{B} := \{P_i\}_{i=1}^{d}$, where $P_i := |i\rangle\langle i|$

$$[X(\mathbb{F}, \mathbb{B}_0)]_{ij} := \mathrm{Tr}\left(F_i P_j^{(0)}\right)$$

$$\boldsymbol{p}_{\mathbb{F}}(\rho) = X(\mathbb{F}, \mathbb{B}_0)\boldsymbol{p}$$

"transition matrix" between two bases is **stochastic**

~~Output probability distributiton is **more uniform** than input one~~

Incompatibility relative to $\mathbb{B}_0$: Generalized Measurements

Notation: we write $\mathbb{F} \succ\!\succ^{\mathbb{B}_0} \mathbb{G}$ if and only if there exists a **stochastic** matrix M such that $X(\mathbb{G}, \mathbb{B}_0) = M X(\mathbb{F}, \mathbb{B}_0)$

"A $\mathbb{G}$ measurement can be emulated by any kind of probabilistic post-processing from an $\mathbb{F}$ measurement"

Incompatibility relative to $\mathbb{B}_0$: Generalized Measurements

Notation: we write $\mathbb{F} \succcurlyeq^{\mathbb{B}_0} \mathbb{G}$ if and only if there exists a **stochastic** matrix M such that $X(\mathbb{G},\mathbb{B}_0) = M X(\mathbb{F},\mathbb{B}_0)$

"A $\mathbb{G}$ measurement can be emulated by any kind of probabilistic post-processing from an $\mathbb{F}$ measurement"

" $\succcurlyeq^{\mathbb{B}_0}$ " is a preorder over **POVMs**

Preorder

$\mathbb{F} \succ^{\mathbb{B}_0} \mathbb{G}$

Monotones

(functions from POVMs to non-negative Real)

$f_{\mathbb{B}_0}(\mathbb{F}) \geq f_{\mathbb{B}_0}(\mathbb{G})$ Compatibility measure

$f_{\mathbb{B}_0}(\mathbb{F}) \leq f_{\mathbb{B}_0}(\mathbb{G})$ Incompatibility measure

Incompatibility relative to $\mathbb{B}_0$: Generalized Measurements

Notation: we write $\mathbb{F} \succ\!\!\succ^{\mathbb{B}_0} \mathbb{G}$ if and only if there exists a **stochastic** matrix M such that $X(\mathbb{G},\mathbb{B}_0) = MX(\mathbb{F},\mathbb{B}_0)$

"A $\mathbb{G}$ measurement can be emulated by any kind of probabilistic post-processing from an $\mathbb{F}$ measurement"

"$\succ\!\!\succ^{\mathbb{B}_0}$" is a preorder over **POVMs**

Preorder

$\mathbb{F} \succ^{\mathbb{B}_0} \mathbb{G}$

Monotones

(functions from POVMs to non-negative Real)

$f_{\mathbb{B}_0}(\mathbb{F}) \geq f_{\mathbb{B}_0}(\mathbb{G})$ Compatibility measure

$f_{\mathbb{B}_0}(\mathbb{F}) \leq f_{\mathbb{B}_0}(\mathbb{G})$ Incompatibility measure

Results:

i) Complete family of monotones: like orthogonal measurements, but *convex* + *homogeneous* functions

ii) Get rid of basis-dependence: Reduces to " $\mathbb{F}$ is a **parent measurement** of $\mathbb{G}$ "

Summary

- We have introduced a notion of **incompatibility for quantum measurements,** relative to a **reference basis**
- Approach yields complete family of monotones, i.e., **quantifiers of incompatibility**
- Connection of **incompatibility**, quantum **coherence** and **uncertainty relations**
- Generalization to **arbitrary POVM measurements**

arXiv: 1901.06382

Thank you!