Iso-entangled Mutually Unbiased Bases and mixed states t–designs

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in collaboration with

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What is this talk about ?

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relevant for the standard **Quantum Theory**,

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relevant for the standard **Quantum Theory**,

for instance:

- **Mutually Unbiased Bases** (**MUB**s)
- **Symmetric Informationally Complete generalized quantum measurements** (**SIC POVM**s)
- Complex **projective t-designs** formed of pure quantum states and their generalizations:
- selected constellations of mixed states which form **mixed states t-designs**.

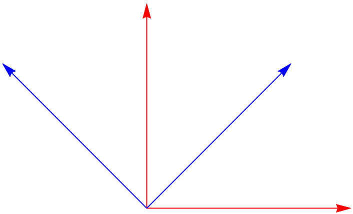
Why we do it ? Beacause we
a) do not fully understand these structures relevant for **quantum theory** !
b) wish to construct novel schemes of **generalized measurements** and
c) design techniques averaging over the set of **density matrices** of size N

Mutually Unbiased Bases I

- Two orthogonal bases consisting of n vectors each in $\mathcal{H}_N$ are called **mutually unbiased** (MUB) if

$$|\langle\phi_i|\psi_j\rangle|^2 = \frac{1}{N} \ , \quad \text{for} \quad i,j = 1,\dots,N \ .$$

- Such bases provide maximally different quantum measurements.
- For a complex Hilbert space of dimension N there exist at most $N+1$ such bases.
- Example $N=2$:
  3 eigenbases of $\sigma_x, \sigma_y, \sigma_y$



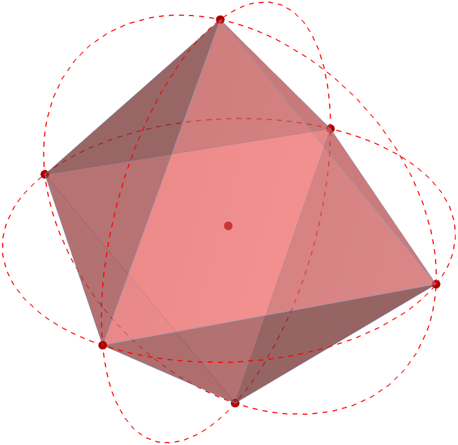
Two unbiased bases in $\mathbb{R}^2$

Mutually Unbiased Bases & Hadamard matrices

- Full sets of $(N+1)$ MUB's are known if dimension is a **power of prime**, $N = p^k$.
  For $\mathbf{N} = \mathbf{6} = 2 \times 3$ only $3 < 7$ MUB's are known!
- A transition matrix $H_{ij} = \langle \phi_i | \psi_j \rangle$ from one **unbiased** basis to another forms a **complex Hadamard** matrix, which is
  a) **unitary**, $H^\dagger = H^{-1}$,
  b) has "**unimodular**" entries, $|H_{ij}|^2 = 1/N, \;\; i,j = 1,\dots,N$.
- Classification of all **complex Hadamard matrices** is complete for $N = 2, 3, 4, 5$ only. (**Haagerup** 1996)
  see Catalog of **complex Hadamard matrices**, at
  http://chaos.if.uj.edu.pl/∼karol/hadamard

Standard set of 2-qubit MUBs

consists of 3 separable bases + 2 maximally entangled bases in $\mathcal{H}_4$



- Reduced states ρ_A and ρ_B form 6 (doubly degenerated) vertices of the regular octahedron within the Bloch ball (eigenvectors of $\sigma_x, \sigma_y, \sigma_z$ $= 3$ MUBs for $N = 2$) and 8-fold degenerated maximally mixed state $\mathbb{1}/2$ in the centre.

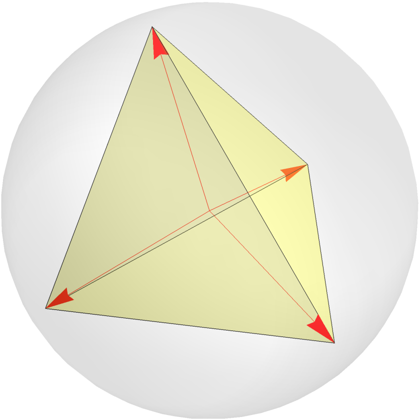
Symmetric Informationally Complete POVM

- Symmetric informationally complete (SIC) POVM is such a set of N^2 vectors $\{|\psi_i\rangle\}$ in $\mathcal{H}_N$, that

$$|\langle\psi_i|\psi_j\rangle|^2 = \frac{1}{N+1}$$

  Zauner (1999), Rennes, Blume-Kohout, Scott, Caves (2003)

- They may be thought as equiangular structures in the Hilbert space.
- SIC POVM are found analitically for $N = 2, \ldots, 24$ and numerically up to 151 + some special cases: $N = 844$ **Grassl & Scott** (2017)



4 pure states at the Bloch sphere forming a SIC for $N = 2$.

Complex projective t-designs

Definition

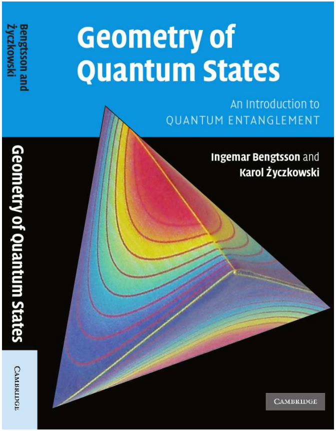
Any ensemble $|\psi_i\rangle_{i=1}^{M}$ of **pure** states in $\mathcal{H}^N$ is called **complex projective t-design** if for any polynomial f_t of degree at **most** t in both components of the states and their conjugates the average over the ensemble coincides with the average over the space $\mathbb{C}P^{N-1}$

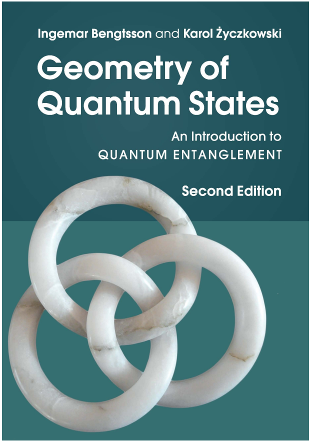
$$\frac{1}{M}\sum_{i=1}^{M} f_t\{\psi_i\} = \int_{\mathbb{C}P^{N-1}} f_t(\psi)\,d\psi_{FS}.$$

with respect to the unitarily–invariant Fubini–Study measure $d\psi_{FS}$.

- **Complex projective** t–designs are used for quantum state tomography, quantum fingerprinting and quantum cryptography.
- Examples of **2-designs** include maximal sets of mutually unbiased bases (MUB) and symmetric informationally complete (SIC) POVM.
- the **larger** t the **better** design approximates the set of states..

To know more about these issues consult the book





Cambridge University Press,
I edition, **2006**

II edition, **2017**
(new chapters on multipartite entanglement & discrete structures in the Hilbert space),

Interesting case – **isoentangled SIC-POVM**

- Averaging property implies a condition for the average entanglement (measured by the purity of partial trace) of vectors in a 2-design in $\mathcal{H}_N \otimes \mathcal{H}_N$

$$\left\langle \mathrm{Tr}\left[(\mathrm{Tr}_A |\psi_i\rangle\langle\psi_i|)^2\right]\right\rangle = \frac{2N}{N^2+1}$$

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- **Zhu & Englert** (2011) found an interesting constelation of $4^2 = 16$ states in $\mathcal{H}_2 \otimes \mathcal{H}_2$ forming a SIC for two-qubit system, such that entanglement of all states is constant,

$$\mathrm{Tr}\left[(\mathrm{Tr}_A |\psi_i\rangle\langle\psi_i|)^2 \right] = \frac{4}{5}, \quad \text{for} \quad i = 1, \ldots, 16.$$

Such a set of states can be obtained from a single *fiducial* state $|\phi_0\rangle$ by **local unitary** operations, $|\phi_j\rangle = U_j \otimes V_j |\phi_0\rangle$.

Isoentangled MUBs for 2 qubits?

Question:

Isoentangled MUBs for 2 qubits?

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Is there a similar configuration for the full set of 5 iso-entangled MUBs for 2 qubits?

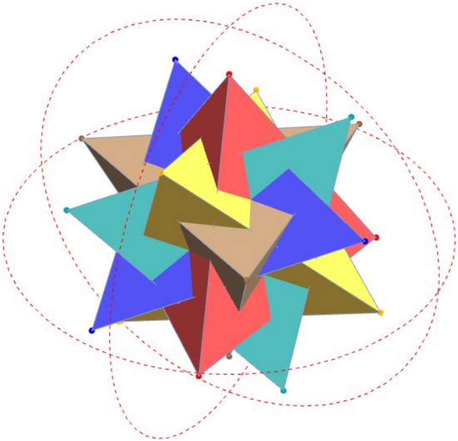
the standard MUB solution for $N = 4$ consists of
3 **separable** bases and 2 **maximally entangled**...

The answer is positive!

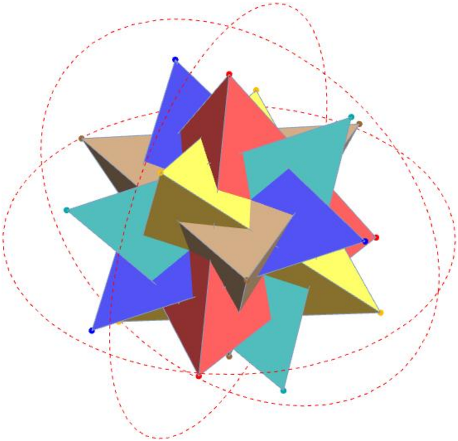
$$|\phi_0\rangle = \frac{1}{20}(a_+ |00\rangle - 10i\,|01\rangle + (8i-6)\,|10\rangle + a_- |11\rangle),$$

where $a_\pm = -7 \pm 3\sqrt{5} + i(1 \pm \sqrt{5})$
and other states are **locally equivalent**, $|\phi_j\rangle = U_j \otimes V_j\,|\phi_0\rangle$

Each of $5 \times 4 = 20$ pure states $|\psi_j\rangle$ in $\mathcal{H}_2 \otimes \mathcal{H}_2$
will be represented by its partial trace, $\rho_j = \mathrm{Tr}_B|\psi_j\rangle\langle\psi_j|$
belonging to the Bloch ball of one-qubit mixed states.



- Each basis is represented by a regular **tetrahedron** inside the Bloch ball.



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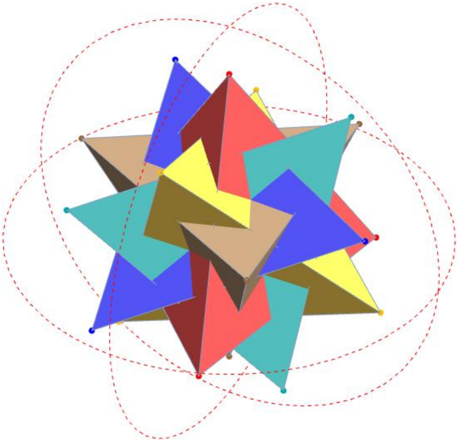
- **Each colour corresponds to a single basis.**



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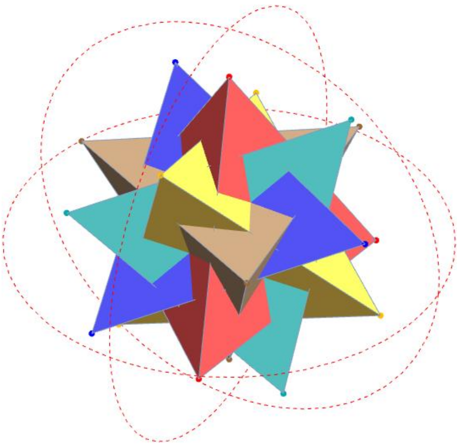

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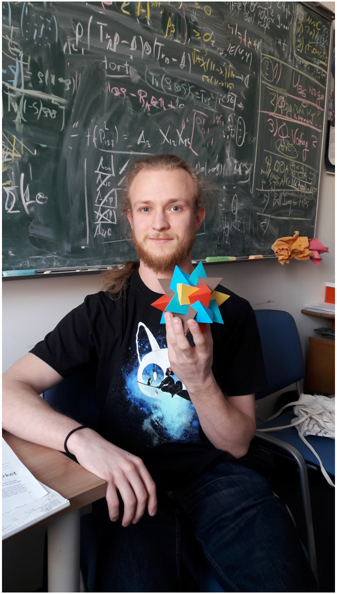
- **Each colour corresponds to a single basis.**
- Entire five-color set forms a **regular 5-tetrahedra compound**.
- Its convex hull forms a

  regular dodecahedron,

  different from the one of

  Zimba and **Penrose**...



Jakub Czartowski and his sculpture

Mixed states t-designs

Generalized designs

In quantum theory one uses

- **projective designs** formed by **pure states**, $|\psi_i\rangle \in \mathcal{H}_N$
- **unitary designs** formed by **unitary matrices**, $U_i \in U(N)$ (which induce designes in the set of maximally entangled states, $|\phi_j\rangle = (U_j \otimes \mathbb{1})|\psi_+\rangle$
- **spherical designs** - sets of points *evenly distributed* at the sphere S^k
- related notions, e.g. **conical designs**, **Graydon** & **Appleby** (2016), mixed designs by **Brandsen, Dall'Arno, Szymusiak** (2016)

These examples for special case of a general construction of **averaging sets** by **Seymour and Zaslavsky (1984)**. It concerns a collection of M points x_j from an arbitrary measurable set Ω with measure μ such that

$$\frac{1}{M}\sum_{i=1}^{M} f_t(x_i) = \int_\Omega f_t(x)d\mu(x),$$

where $f_t(x)$ denote selected continuous functions, e.g. $f_t(x) = x^t$.

Mixed states t-designs

We apply this idea for a compact set of mixed states $\Omega_N \subset \mathbb{R}^{N^2-1}$ endowed with the flat Hilbert-Schmidt measure $\mathrm{d}\rho_{HS}$

Definition

Any ensemble $\{\rho_i\}_{i=1}^{M}$ of M density matrices of size N is called a **mixed states t-design** if for any polynomial g_t of degree t in the eigenvalues λ_j of the state ρ the average over the ensemble is equal to the mean value over the space of mixed states Ω_N with respect to the **Hilbert-Schmidt** measure $\mathrm{d}\rho_{HS}$,

$$\frac{1}{M}\sum_{i=1}^{M} g_t(\rho_i) = \int_{\Omega_N} g_t(\rho)\,\mathrm{d}\rho_{HS}\,. \tag{1}$$

Method of genereting mixed states t-designs

Proposition 1.

Any complex **projective t-design** $\{|\psi_i\rangle\}_{i=1}^{M}$
in the composite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ of size $N \times N$
induces, by partial trace,
a **mixed states design** $\{\rho_i\}_{i=1}^{M}$ in Ω_N with $\rho_i = \mathrm{Tr}_B\, |\psi_i\rangle\langle\psi_i|$.

The same property holds also for the dual set $\{\rho_i' : \rho_i' = \mathrm{Tr}_A\, |\psi_i\rangle\langle\psi_i|\}$.

Proof is based on the fact that **Fubini–Study measure** in the space of pure states od size N^2 induces, by partial trace, the flat **HS measure** in the space Ω_N of mixed states of size N.

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Observation 1.

Every positive operator-valued measurement (POVM)
induces a mixed states **1-design**,
as its barycenter coincides with the maximally mixed state $\mathbb{1}/N$.

If and only if conditions for mixed states t-designs

Proposition 2.

A set $\{\rho_j\}_{j=1}^M$ of density matrices of size N forms a **mixed states t-design** **if and only if** it saturates the inequality *analogous to the Welsh bound* – Scott (2006)

$$2\,\mathrm{Tr}\left(\frac{1}{M}\sum_{i=1}^{M}\rho_i^{\otimes t}\int_{\Omega_N}\rho^{\otimes t}\,\mathrm{d}\rho_{HS}\right)-\frac{1}{M^2}\sum_{i,j=1}^{M}\mathrm{Tr}(\rho_i\rho_j)^t\leq\gamma_{N,t}$$

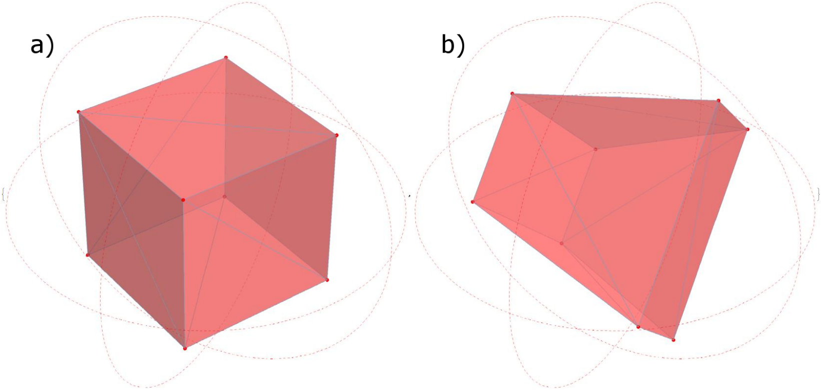
where $\gamma_{N,t} := \mathrm{Tr}\,\omega_{N,t}^2$ and $\omega_{N,t} := \int_{\Omega_N}\rho^{\otimes t}\,\mathrm{d}\rho_{HS}$

Observation 2.

Due to the theorem of Seymour and Zaslavsky
mixed–states t-designs exists for any order t and matrix size N.

Isoentangled 2-qubit SIC-POVM formed of 16 pure states[1]

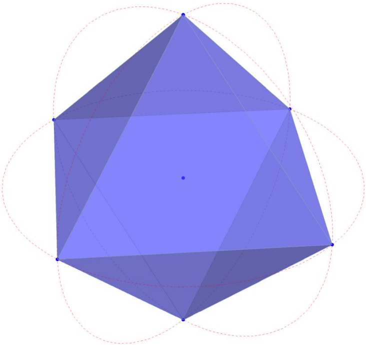
both partial traces form a constelation of 8 (doubly degenerated) points inside Bloch ball



- In Alice reduction SIC-POVM yields a Platonic solid - the cube. The constellation in the reduction of Bob is not as regular as for Alice.

[1] **Zhu & Englert (2011)**

Hoggar example[2] of 60 states in $\mathcal{H}_4$



- Hoggar provides an example (no. 24) of projective 3-design in $\mathcal{H}_4$ attained by considering particular complex polytope that consists of 60 states.

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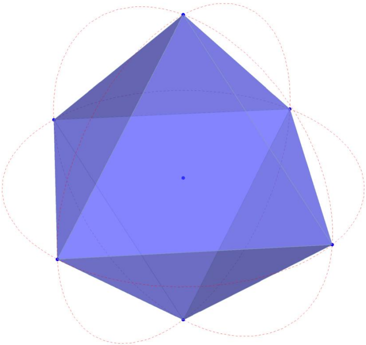

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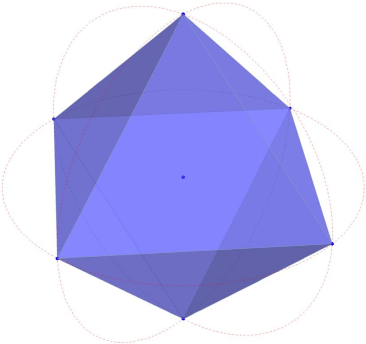

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- both reductions yield the same structure inside the Bloch ball as the one generated by the standard MUB for 2 qubits.
- This implies that reducing 20 states forming the standard set of **MUBs for 2 qubits** induces **mixed 3-design**.

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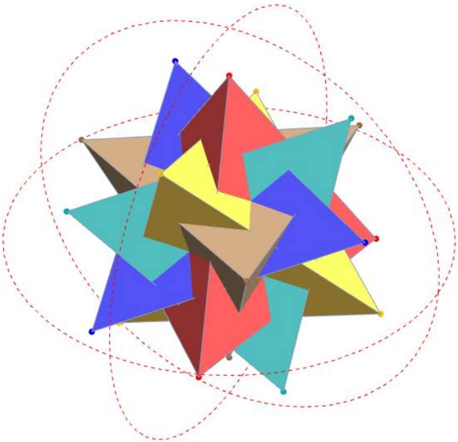


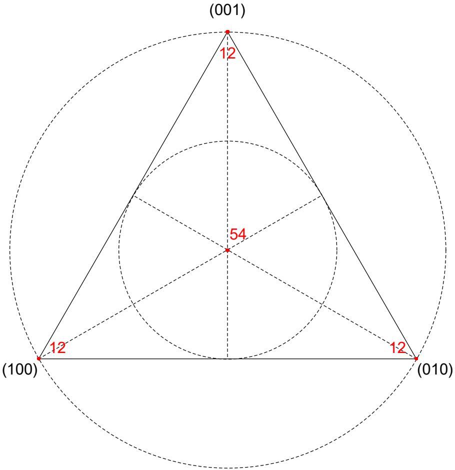
- $5 \times 4 = 20$ mixed states obtained by partial trace, $\rho_j = \mathrm{Tr}_B |\psi_j\rangle\langle\psi_j|$ of 20 pure states $|\psi_j\rangle$ from **isoentnagled MUB** in $\mathcal{H}_2 \otimes \mathcal{H}_2$ form a mixed states **2-design** inside the Bloch ball.

  In fact they form a **3-design** !

Qutrits

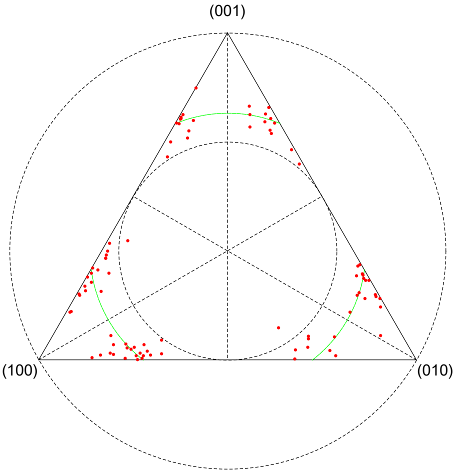
What can be said about analogous configurations for qutrits?

Standard representation of 2 qutrits MUB



- Standard representation of MUB for 2 qutrits consists of 90 states which form 10 bases: 4 separable and 6 maximally entangled.
- Representation of density matrices in the triangle of eigenvalues: three (12-fold degenerated) corners + the center of degeneration $6 \times 9 = 54$.

Numerical results for 2 qutrits isoentangled MUB



- Numerical search for isoentangled MUB for 2 q-trits were inconclusive.
- However, all the points seem to converge at the green circle - subset of density matrices with proper purity.

Example: t-designs in the interval (= averaging sets)

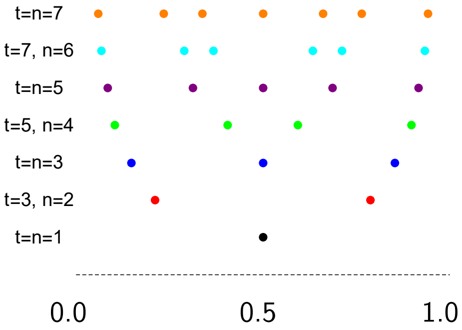
- Consider a measure $\mu(x)$ defined on the interval $[0,1]$ and a minimal sequence of points $\{x_i : x_i \in [0,1]\}_{i=1}^{M}$ such that

$$\frac{1}{M}\sum_{i=1}^{M} x_i^t = \int_0^1 x^t \mu(x)\,\mathrm{d}x\,. \qquad (2)$$

- Such structures may find use in approximate integration using **Taylor expansion**

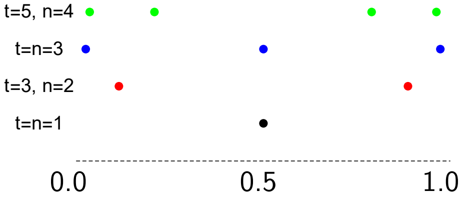
$$\int_0^1 f(x)\,\mathrm{d}x = \left(\sum_{i=0}^{t}\sum_{j=1}^{M}\frac{1}{i!}\frac{\mathrm{d}^i f(x)}{\mathrm{d}x^i}\bigg|_{x=x_0}(x_j - x_0)^t\right) + O(x^{t+1}) \qquad (3)$$

Lebesgue measure on an interval



- $\mu(x) = 1$ defines flat measure.
- Configurations have been found up to $t = 7$.

HS measure: a single-qubit example

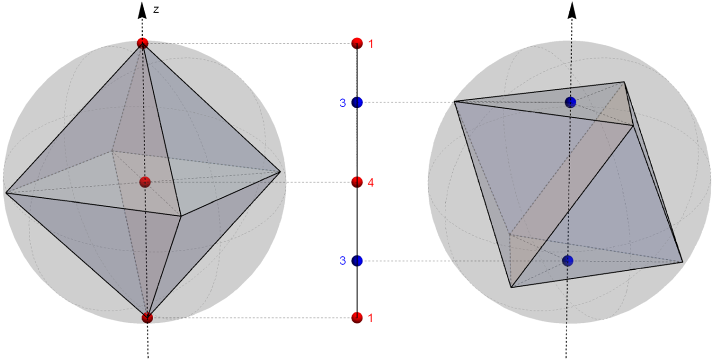


- Consider the Hilbert-Schmidt measure on eigenvalues of density matrices,
  $$P(\lambda_1, \lambda_2) \sim (\lambda_1 - \lambda_2)^2$$
  which leads to the **flat** measure inside the Bloch Ball
  $$\mu_{HS}(x) = 3(2x - 1)^2$$
  with radius $r = |2x - 1|$
- For $t = 5$ we found $N = 4$ points.

Projection of projective designs onto the simplex I

Pure states t–design $\{|\psi_j\rangle\}$ in $\mathcal{H}_N$ cover the set of **pure** states. Their projection on the simplex due to **decoherence**, $\mathbf{p}_j = \mathrm{diag}(|\psi_j\rangle\langle\psi_j|)$ gives a t–design in the simplex Δ_N according to the flat measure:

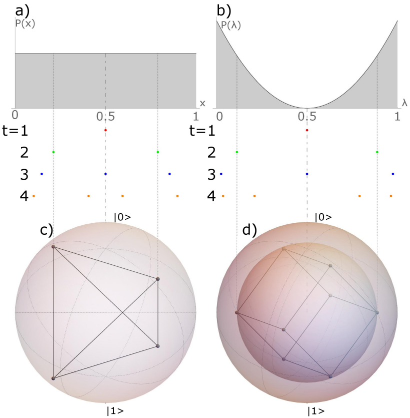
(example for $N = t = 2$ and **Bloch sphere**).



Approximate **integration** rules:

Simpson $1:4:1$ **Gauss–Legendre** $3:3 = 1:1$

Projection of quantum states designs onto the simplex II



- a) Pure states t–design $\{|\psi_j\rangle\}$ in $\mathcal{H}_N$ cover the set of **pure** states. Their projection on the simplex due to **decoherence**, $\mathbf{p}_j = \mathrm{diag}(|\psi_j\rangle\langle\psi_j|)$ gives a t–design in the simplex Δ_N according to the flat measure: (example for $N = t = 2$ and the **Bloch sphere**).
- b) Mixed states t–design $\{\rho_j\}$ cover the set Ω_N of **mixed** states. Their projection on the simplex related to **spectrum**, $\mathbf{p}_j = \mathrm{eig}(\rho_j)$, gives a t–design in the simplex Δ_N according to the HS measure: (example for $N = t = 2$ and the **Bloch ball**).

Concluding Remarks

- Configuration of 20 **pure** states in $\mathcal{H}_4$ which form the full set of 5 **iso-entangled MUBs** for 2 qubits is constructed.
- Notion of **mixed states t-design** is introduced and **necessary and sufficient** conditions for $\{\rho_j\}$ to be a design are established.
- **Projective** t-designs on composite spaces $\mathcal{H}_N \otimes \mathcal{H}_N$ induce, by partial trace, **mixed states** t-designs in the set Ω_N of mixed states.
- **Simplicial** t**–designs** in the simplex Δ_n obtained from projective t–designs $\{|\psi_j\rangle\}$ in $\mathcal{H}_N$ by **decoherence**, $\mathbf{p}_j = \mathrm{diag}(|\psi_j\rangle\langle\psi_j|)$.
- One qubit examples of mixed states t-designs form regular structures *inside* the **Bloch ball**.

- *Open questions*
  - Analytic isoentangled structure for a qutrit-qutrit system?
  - Minimal size M of mixed states t–design in Ω_N?