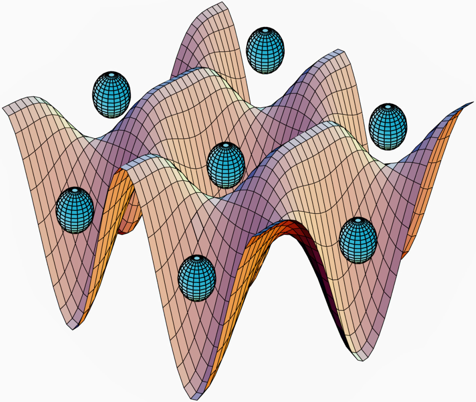
Driven Quantum Dynamics: Will It Blend?

Leonardo Banchi (University of Florence, Italy)

Question



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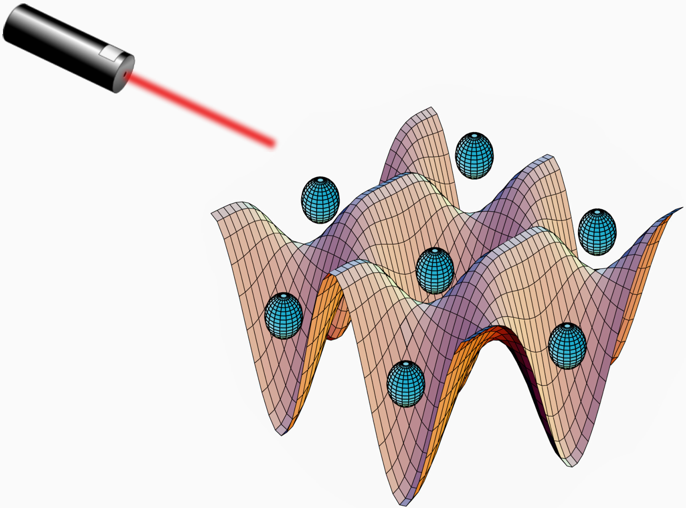

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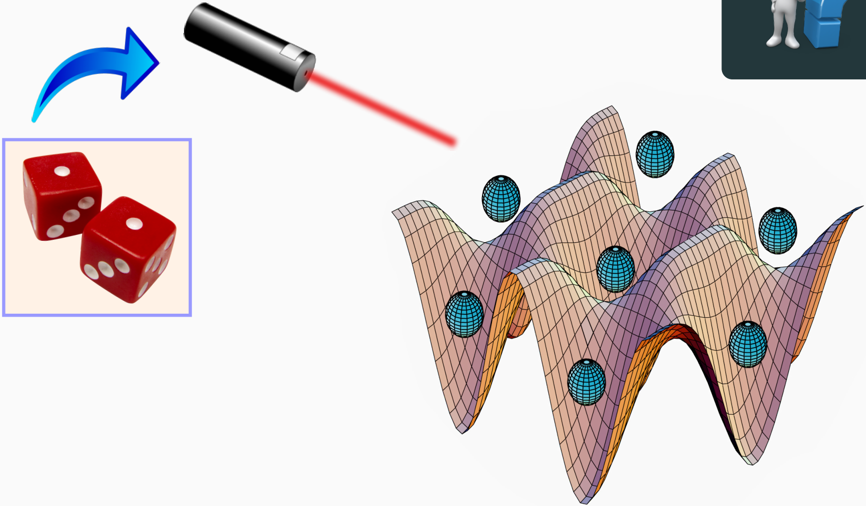

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A driven quantum system

$$H(t) = H_0 + g(t)V$$

Resulting unitary operation after a time T

$$U = \mathcal{T} e^{-i \int_0^T H(t)\, dt}$$

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L. Banchi, D. Burgarth, M. J. Kastoryano, Phys. Rev. X 7, 041015 (2017)

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What if $g(t)$ is random?

We show (under some conditions):

- After some time $\{U\}$ is fully random (Haar)
- Estimation of the blending time using the theory of open quantum systems and many-body techniques (Bethe Ansatz)

L. Banchi, D. Burgarth, M. J. Kastoryano, Phys. Rev. X 7, 041015 (2017)

Motivation

Very useful in quantum information processing

- Quantum encryption

$$|\psi_1\rangle\,,\ |\psi_2\rangle\,,\ \ldots \qquad \longmapsto \qquad U_1\,|\psi_1\rangle\,,\ U_2\,|\psi_2\rangle\,,\ \ldots$$

- State tomography (2-design)
    - Optical tomography with $2N$-designs (N = number of photons)

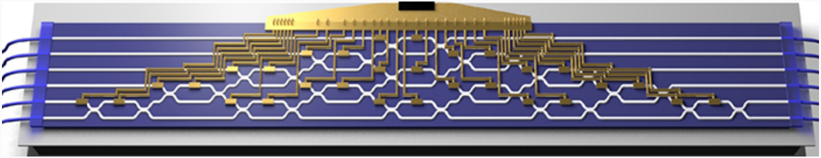
Banchi, Kolthammer, Kim, PRL 2018

- Noise estimation in open quantum systems
  (apply random unitaries such that the coherent part is averaged out)

- Generation of highly entangled states

Quantum Supremacy

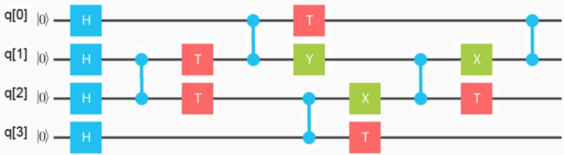
credit: J. Carolan et al./Science 2015



Boson sampling experiments require sampling from hard quantum distribution

Common approach

Theoretical scheme:



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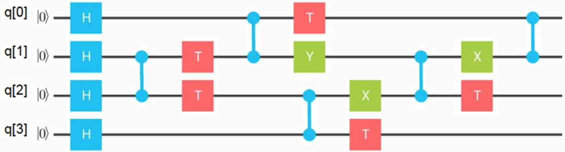


Physics: courtesy of S. Boixo (Google)

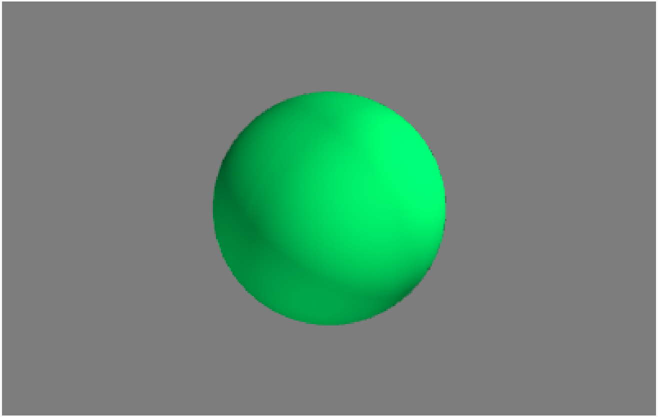
$$\hat{H}_{BH}(t) = \sum_{j=1}^{N} \left[\delta_j(t)\hat{n}_j + \frac{\eta}{2}\hat{n}_j(\hat{n}_j - 1) + i\left(\hat{a}_j e^{i\phi(t)} - \hat{a}_j^\dagger e^{-i\phi(t)}\right) F_j(t) \right] + \sum_{j=1}^{N-1} g(t)\left(\hat{a}_j \hat{a}_{j+1}^\dagger + \hat{a}_{j+1}\hat{a}_j^\dagger\right)$$

detuning	microwave amplitude	microwave phase	g-pulse	anharmonicity
$\delta_j(t)$	$F_j(t)$	$\phi_j(t)$	$g(t)$	η

;

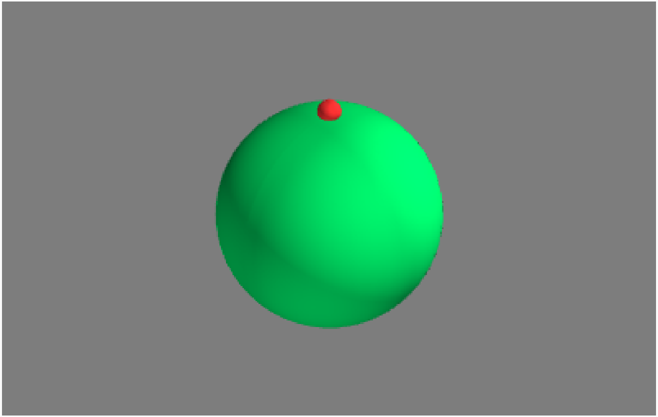
How?

Geometric picture



$$U = \mathcal{T} e^{-i \int_0^T H(t)\, dt}$$

Geometric picture



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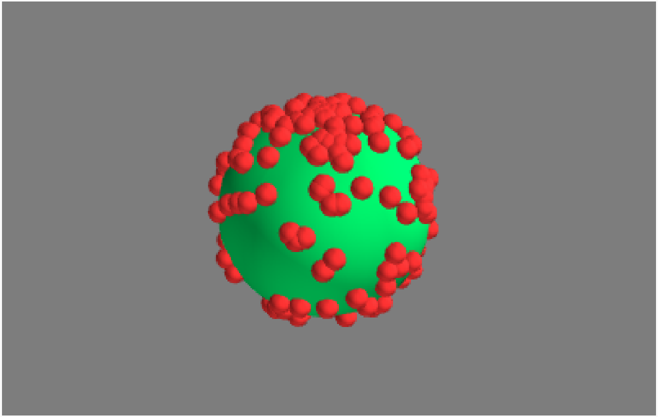

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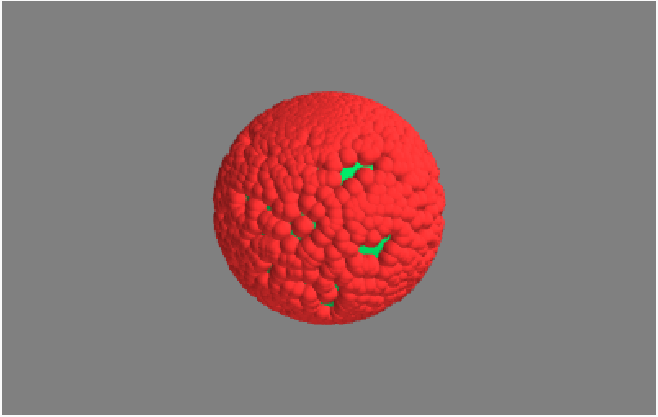

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How “random” is the evolution?

Comparing probability distributions on unitaries as a physical process

$$\left\| \mathbb{E}_U \left[U\rho U^\dagger \right] - \int U\rho U^\dagger \, \mu_{\mathrm{Haar}}(dU) \right\| < \epsilon$$

How "random" is the evolution?

Comparing probability distributions on unitaries as a physical process

$$\left\| \mathbb{E}_U \left[U^{\otimes q} \, \rho \, U^{\otimes q \dagger} \right] - \int U^{\otimes q} \, \rho \, U^{\otimes q \dagger} \, \mu_{\mathrm{Haar}}(dU) \right\|_{?} < \epsilon$$

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q-design

$$\left\| \mathbb{E}_U \left[U^{\otimes q} \, \rho \, U^{\otimes q \dagger} \right] - \int U^{\otimes q} \, \rho \, U^{\otimes q \dagger} \, \mu_{\mathrm{Haar}}(dU) \right\|_{\diamond} < \epsilon$$

No single (global) measurement can distinguish between the two processes with probability larger than ϵ

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Comparing probability distributions on unitaries as a physical process

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$$\left\| \mathbb{E}_U \left[U^{\otimes q} \, \rho \, U^{\otimes q \dagger} \right] - \int U^{\otimes q} \, \rho \, U^{\otimes q \dagger} \, \mu_{\mathrm{Haar}}(dU) \right\|_{\diamond} < \epsilon$$

Using vectorisation

$$X = \sum_{ij} X_{ij} \left| i \right\rangle \left\langle j \right| \quad \mapsto \quad |X\rangle\rangle = \sum_{ij} X_{ij} \left| ij \right\rangle$$

$|AX\rangle\rangle = A \otimes 1 |X\rangle\rangle$ and $|XA\rangle\rangle = 1 \otimes A^T |X\rangle\rangle$. With $U^{\otimes q,q} = U^{\otimes q} \otimes (U^{\otimes q})^*$

Expanders (weaker)

$$e(\mu_U, q) = \left\| \mathbb{E}_U \left[U^{\otimes q,q} \right] - \int U^{\otimes q,q} \, \mu_{\mathrm{Haar}}(dU) \right\|_{\infty} < \epsilon$$

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Operational way of measuring if two probability distributions are “close enough”

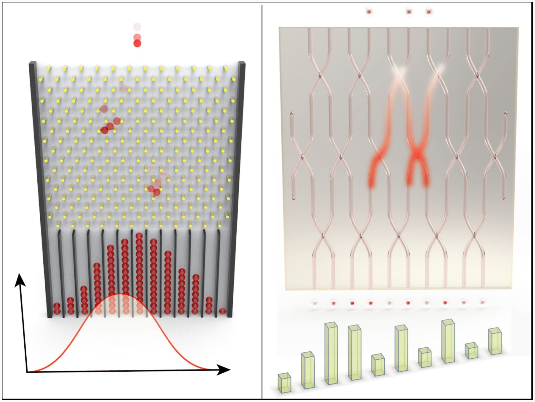
$$e(\mu_U, q) \leq 2q \, \mathcal{W}(\mu_U, \mu_{\mathrm{Haar}}) \qquad \text{Brandao et al. (2016)}$$

Physical picture

In boson sampling experiments the photon distribution follows

$$|\mathrm{per}(\tilde{U})|^2 = \sum_{\sigma,\sigma'} \prod_{i,j=1}^{q} \tilde{U}_{i,\sigma(i)} \tilde{U}^*_{j,\sigma'(j)} = \mathrm{Tr}\left[U^{\otimes q,q} \mathcal{K}_{\mathrm{b.s.}}\right]$$

q is the number of injected photons – "quantum Plinko machine"

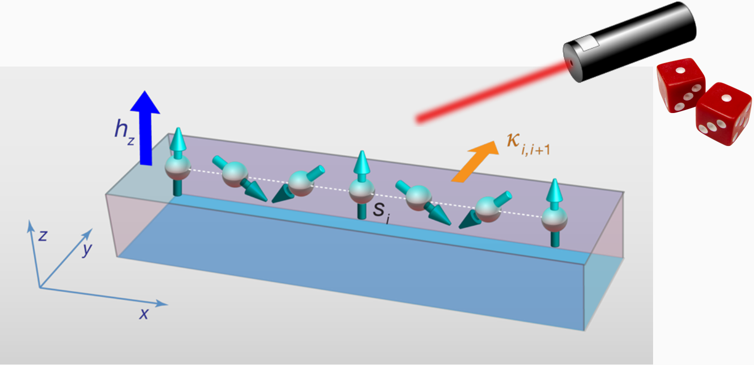


Physical picture

Convergence of spin correlations in randomly driven XY spin chain

$$\langle S_i^\alpha(t) S_{i+q}^\beta(t) \rangle = \mathrm{Tr}\left[U^{\otimes q,q} \mathcal{K}_{\mathrm{XY}} \right] \qquad \text{for } \alpha, \beta \in \{x, y\}$$

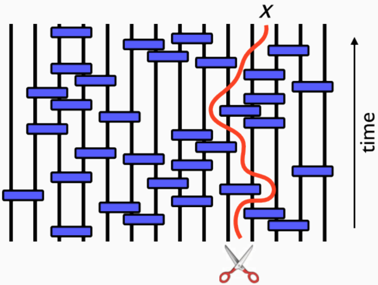
q is the distance between spins



Physical picture

Renyi entropies are studied in information scrambling

$$S_q = \frac{1}{1-q}\,\mathrm{Tr}[\rho^q] \equiv \frac{1}{1-q}\,\mathrm{Tr}[(\rho\otimes\rho\otimes\cdots\otimes\rho)\,\mathcal{P}]$$



If $\rho = \mathrm{Tr}_{\mathrm{ancilla}}[U\,|\psi\rangle\langle\psi|\,U^\dagger]$ then $\mathbb{E}[\mathrm{Tr}(\rho^q)] = \mathrm{Tr}\,[U^{\otimes q,q}\mathcal{K}_{\mathrm{Renyi}}]$

Results

Random pulse

Stochastic driving of a quantum system

$$\hat{H}(t) = H + g(t)V$$

e.g. with random amplitudes and phases

$$g(t) = \sum_{k=1}^{K} A_k \cos(\omega_k t + \varphi_k)$$

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The average is over the random amplitudes, phases etc.

$$\mathbb{E}_U\left[U^{\otimes q} \rho U^{\otimes q \dagger}\right] = \mathbb{E}\left[\left(\mathcal{T} \mathrm{e}^{-\mathrm{i} \int_0^T \hat{H}(s)\, ds}\right)^{\otimes q} \rho \left(\mathcal{T} \mathrm{e}^{\mathrm{i} \int_0^T \hat{H}(s)\, ds}\right)^{\otimes q}\right]$$

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Simplification when $g(t)$ is

- Gaussian (central limit theorem)
- Harmonic $\mathbb{E}[g(t+s)g(t)] = c(s)$
- Short correlations $Tc(Ts) \simeq \frac{\sigma}{2}\delta(s)$

Simplifications

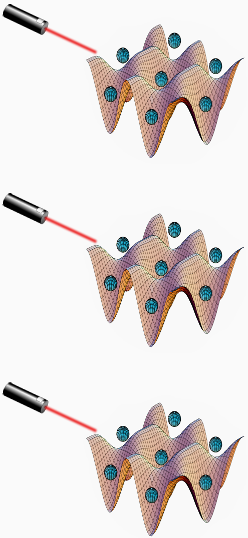
$$
\begin{aligned}
\mathbb{E}_U\left[U^{\otimes q}\rho U^{\otimes q\dagger}\right] &= \mathbb{E}\left[\left(\mathcal{T}e^{-\mathrm{i}\int_0^T \hat{H}(s)\,ds}\right)^{\otimes q}\ \rho\ \left(\mathcal{T}e^{\mathrm{i}\int_0^T \hat{H}(s)\,ds}\right)^{\otimes q}\right] \\
&\simeq e^{-T\mathcal{L}^q}\ \rho\ ,
\end{aligned}
$$

where

$$
\mathcal{L}^q\rho = -\mathrm{i}\left[\mathrm{H}^{\oplus \mathrm{q}}, \rho\right] - \frac{\sigma}{2}\left[\mathrm{V}^{\oplus \mathrm{q}}, \left[\mathrm{V}^{\oplus \mathrm{q}}, \rho\right]\right]
$$

$X^{\oplus q}{=}X{\oplus}X{\oplus}\ldots$, where $\oplus$ is the Kronecker sum $X{\oplus}Y{=}X{\otimes}1{+}1{\otimes}Y$

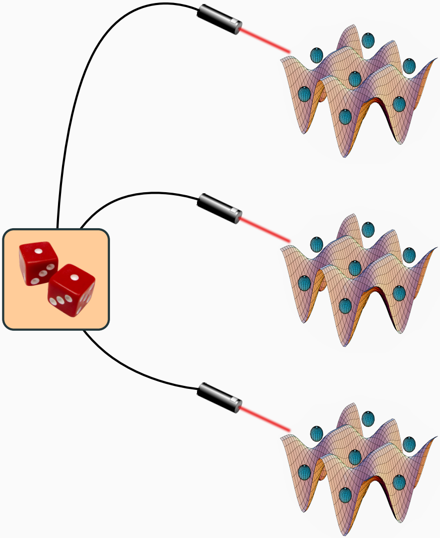
Cartoon picture



Each "replica" is initially

decoupled from the others

Cartoon picture



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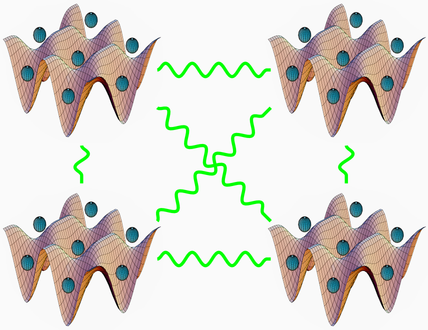
Cartoon picture

After the average over the random pulses these copies are interacting (*à la replica trick*), but in a dissipative way

$$\mathcal{L}^q \rho = -\mathrm{i}\left[\mathrm{H}^{\oplus \mathrm{q}}, \rho\right] - \frac{\sigma}{2}\left[\mathrm{V}^{\oplus \mathrm{q}}, \left[\mathrm{V}^{\oplus \mathrm{q}}, \rho\right]\right]$$



shown for $q = 4$

Controllability implies uniform blending

First central result

If $H + g(t)V$ is **fully-controllable**, then

$$\lim_{t\to\infty} e^{t\mathcal{L}^q}\rho = \int_{\mathrm{Haar}} U^{\otimes q}\rho U^{\otimes q\,\dagger}\, dU$$

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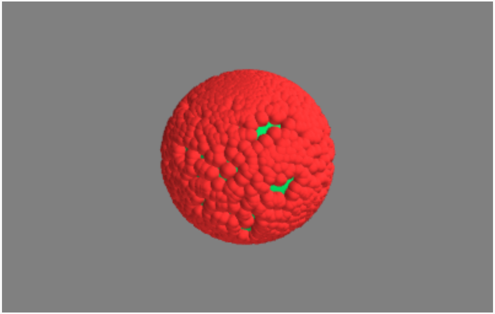

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Proof idea:

fully controllable means, H, V, $[H,V]$, $[H,[H,V]]$, $\dots$ generate the full Lie algebra SU(d)

Schur Weyl duality: $(\mathbb{C}^d)^{\otimes q} = \otimes_\lambda \mathcal{P}^\lambda \otimes U^\lambda$
$\mathcal{P}^\lambda$ irreducible representation of the symmetric group S_q
$\mathcal{U}^\lambda$ irreducible representation of SU(d)

Convergence time

$$\left\| e^{t\mathcal{L}^q} - \lim_{t\to\infty} e^{t\mathcal{L}^q} \right\|_\eta \lesssim A_\eta \, e^{-\lambda^* t}$$

λ^* is the Liouvillean gap, the eigenvalue of $-\mathcal{L}^q$ with minimal non-zero real part

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How do we estimate λ^*?

Complicated problem:

- $\mathcal{L}^q$ is formed by q interacting copies of the original Hilbert space
- Huge Hilbert space
- Restriction to low q is not enough

Many body theory

Let's write the Liouvillean using "vectorised" notation

$$\mathcal{L}_q = -i(H^{\otimes q} \otimes 1 - 1 \otimes H^{\otimes q}) - \frac{\sigma}{2}(V^{\otimes q} \otimes 1 - 1 \otimes V^{\otimes q})^2$$

Second quantized notation: for any operator H

$$H^{\otimes q} = \sum_{ij,u} H_{ij} a_{iu}^{\dagger} a_{ju}$$

q now is the total number of "virtual" particles

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Hubbard-like model (non Hermitean)

$$\begin{aligned}\mathcal{L}_q = &-i \sum_{\alpha\beta u} H_{\alpha\beta}(a^\dagger_{\alpha u\uparrow} a_{\beta u\uparrow} - a^\dagger_{\beta u\downarrow} a_{\alpha u\downarrow}) \\ &- \frac{\sigma}{2} \sum_{\alpha\beta uv} V_{\alpha\alpha} V_{\beta\beta}(n_{\alpha u\uparrow} - n_{\alpha u\downarrow})(n_{\beta v\uparrow} - n_{\beta v\downarrow})\end{aligned}$$

where $n_x = a^\dagger_x a_x$ and V is diagonal.

Mean field predictions

Mean field solution

The gap λ^* is independent on q

- Confirmed in “typical” numerical simulations
- Powerful result: the convergence time the same for all the moments[1]
- Validity because “everything interacts with everything”

[1] For certain choices of the norm...

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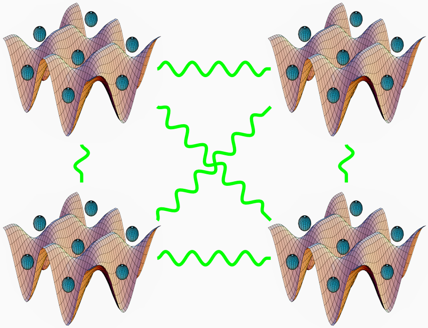

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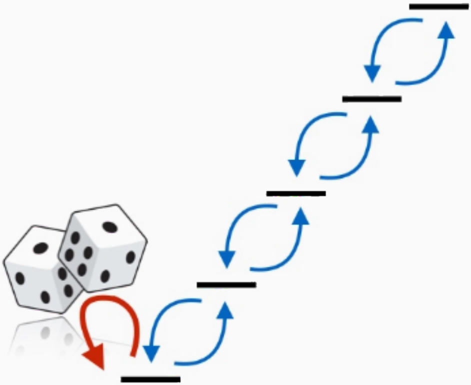
But...

we found (uncommon) **counterexamples**

- (replica?) symmetry breaking in tensor powers

[1] For certain choices of the norm...

Exactly solvable model: symmetric case



$$\mathcal{L}_{\boldsymbol{q}} = -\mathrm{i}\sum_{\alpha}(\mathrm{a}^{\dagger}_{\alpha\uparrow}\mathrm{a}_{\alpha+1,\uparrow} - \mathrm{a}^{\dagger}_{\alpha\downarrow}\mathrm{a}_{\alpha+1,\downarrow} + \mathrm{h.c.}) - \frac{\sigma}{2}(\mathrm{n}^{\uparrow}_{1} - \mathrm{n}^{\downarrow}_{1})(\mathrm{n}^{\uparrow}_{1} - \mathrm{n}^{\downarrow}_{1})$$

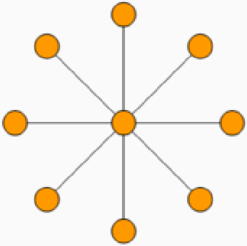
Exactly solvable model: symmetric case

Strong driving limit $\sigma \gg 1$

$$K_i^+ = \tilde{a}_{i\uparrow}^\dagger \tilde{a}_{i\downarrow}^\dagger$$
$$K_i^- = (K_i^+)^\dagger$$
$$K_i^z = (\tilde{n}_{i\uparrow} + \tilde{n}_{i\downarrow} + 1)/2$$

SU(1,1) Richardson-Gaudin model

$$\hat{\mathcal{L}}_q = \frac{2}{\sigma} - \frac{8}{\sigma} \sum_{k=1}^{L-1} g_k \, K_0 \cdot K_k$$



Exactly solvable model: symmetric case

Spectrum from Bethe Ansatz

$$\lambda = -\frac{2}{\sigma}\left(\sum_k g_k n_k + 4\sum_\alpha \frac{1}{\omega_\alpha}\right)$$

n_k is the number of unpaired particles in mode k

$$\sum_k \frac{n_k+1}{\omega_\alpha - 2g_k^{-1}} + \frac{1}{\omega_k} + 2\sum_{\beta\neq\alpha}\frac{1}{\omega_\alpha-\omega_\beta} = 0$$

Solutions related to the roots of Heine-Stieltjes polynomials, so $2g_{k+1}^{-1} < \omega_\alpha < 2g_k^{-1}$

The gap is made with unpaired particles

$$\text{gap} \equiv \lambda^* = \frac{8}{\sigma L}\sin^2\left(\frac{\pi}{L}\right) = \mathcal{O}(L^{-3})$$

L is the length. The gap is independent on q

Exactly solvable model: other cases

Fermionic representation

$S_j^- = \tilde{a}_{j\uparrow}\tilde{a}_{j\downarrow}$
$S_j^+ = (S_j^-)^\dagger$
$S_j^z = (\tilde{a}_{j\uparrow}^\dagger \tilde{a}_{j\uparrow} + \tilde{a}_{j\downarrow}^\dagger \tilde{a}_{j\downarrow} - 1)/2$

SU(2) Gaudin model

$$\hat{\mathcal{L}}_q = -\frac{2}{\sigma} + \frac{8}{\sigma}\sum_{k=1}^{L-1} g_k\, S_0 \cdot S_k$$

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Generic case

$$X^{(j)}_{(x,\uparrow),(y,\uparrow)} = \frac{\tilde{a}^\dagger_{jx\uparrow}\tilde{a}_{jy\uparrow} - \tilde{a}_{jy\uparrow}\tilde{a}^\dagger_{jx\uparrow}}{2} \ ,$$
$$X^{(j)}_{(x,\downarrow),(y,\downarrow)} = \frac{\tilde{a}_{jx\downarrow}\tilde{a}^\dagger_{jy\downarrow} - \tilde{a}^\dagger_{jy\downarrow}\tilde{a}_{jx\downarrow}}{2} \ ,$$
$$X^{(j)}_{(x,\uparrow),(y,\downarrow)} = \tilde{a}^\dagger_{jx\uparrow} W \tilde{a}^\dagger_{jy\downarrow} \ ,$$
$$X^{(j)}_{(x,\downarrow),(y,\uparrow)} = \tilde{a}_{jx\downarrow} W \tilde{a}_{jy\uparrow} \ .$$

SU(2q) Gaudin model

$$\hat{\mathcal{L}}_q = -\frac{2q}{\sigma} + \frac{4}{\sigma}\sum_{k=1}^{L-1} g_k \sum_{\alpha\beta} X^{(0)}_{\alpha\beta} X^{(k)}_{\beta\alpha} \ ,$$

Exactly solvable model: generic cases

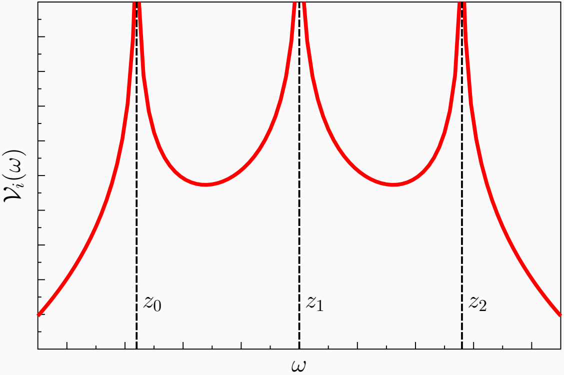
Spectrum from Bethe Ansatz

$$\lambda = -\frac{2}{\sigma}\left[\sum_{k=1}^{L-1} g_k\left(n_{\downarrow k} + n_{\uparrow k}\right) + 4\sum_{\alpha}\frac{1}{\omega_{q,\alpha}}\right]$$

n_k is the number of unpaired particles in mode k

$$\sum_{\beta}\frac{2}{\omega_{j,\beta}-\omega_{j,\alpha}} = \sum_{k=0}^{L-1}\frac{\mu_j^k}{z_k-\omega_{j,\alpha}} + \sum_{\beta,\pm}\frac{1}{\omega_{j\pm1,\beta}-\omega_{j,\alpha}}$$

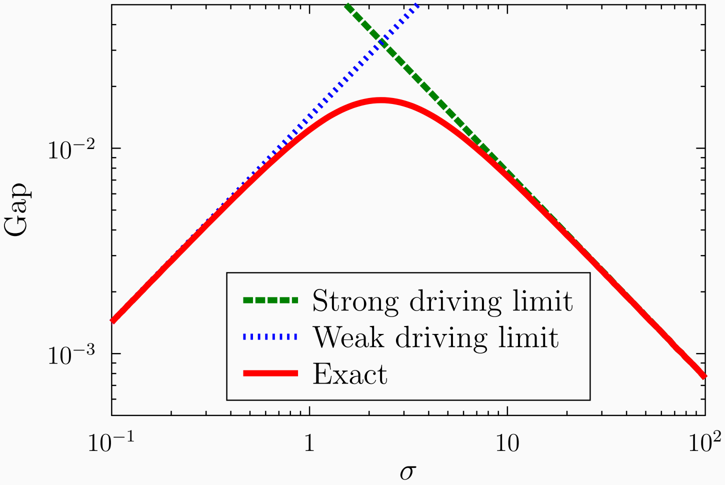
Exactly solvable model: generic cases



Final result

- Gap independent on q
- Mean field analysis is rigorous

Numerical check



Finally?

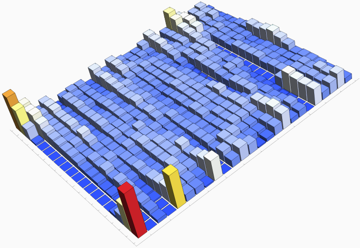
Putting things together

- We need full controllability
- All the moments (typically) converge at the same time
- Open quantum system theory to estimate this time

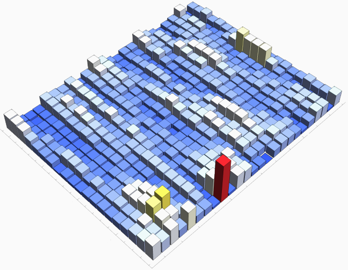
Numerical tests with stochastic pulse

$$g(t) = \sum_{k=1}^{K} A_k \cos(\omega_k t + \varphi_k)$$

Non controllable case



Controllable, but short time

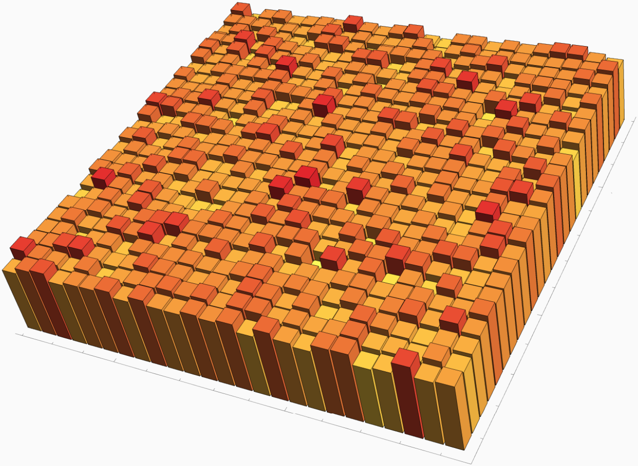


Angle decomposition

$$dU(\varphi_1, \ldots, \varphi_{L^2}) = \prod_{j=1}^{L^2} d\varphi_j \ ,$$

Numerical tests with stochastic pulse

Fully controllable case, after the blending time



Conclusions

Answered this question

When, and how rapidly, a quantum system subject to dynamical noise produces a fully-random (i.e. Haar-uniform) distribution of unitary evolutions?

Different tools from

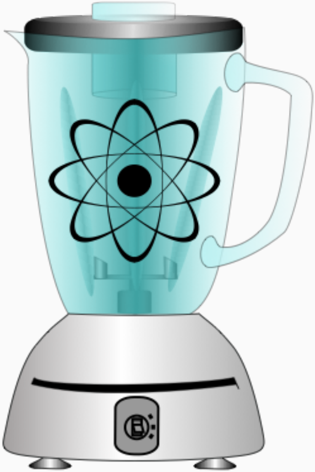
- quantum information (quantum control, q-design)
- open quantum systems (dynamical semigroup, “low energy” Liouvilleans)
- condensed matter physics (Bethe ansatz, mean field in replica space)

Explicit applications:

- Boson sampling experiments
- Estimation of the control time
- Entanglement generation in many-body settings

L. Banchi, D. Burgarth, M. J. Kastoryano, Phys. Rev. X 7, 041015 (2017)

Questions?



Yes, it blends!